6.4 Exponential Growth And Decay

DEFINITION **Separable Differential Equation**

A differential equation of the form $dy/dx = f(y)g(x)$ is called **separable.** We **separate the variables** by writing it in the form

$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx.$$

The solution is found by antidifferentiating each side with respect to its thusly isolated variable.

At this point, we learned how we can integrate both sides, kinda like algebra. We can also separate the variables and integrate each of em!

EXAMPLE 1 Solving by Separation of Variables

Solve for y if $dy/dx = (xy)^2$ and $y = 1$ when $x = 1$.

SOLUTION

The equation is separable because it can be written in the form $dy/dx = y^2x^2$, where $f(y) = y^2$ and $g(x) = x^2$. We separate the variables and antidifferentiate as follows.

$$\cancel{dx}\,\frac{\frac{dy}{\cancel{dx}}}{y^2} = \frac{y^2x^2}{y^2}$$

$$y^{-2}\,dy = x^2\,dx \qquad \text{Separate the variables.}$$

$$\int y^{-2}\,dy = \int x^2\,dx \qquad \text{Prepare to antidifferentiate.}$$

$$-y^{-1} = \frac{x^3}{3} + C \qquad \text{Note that only one constant is needed.}$$

We then apply the initial condition to find C.

$$-1 = \frac{1}{3} + C \Rightarrow C = -\frac{4}{3}$$

$$-y^{-1} = \frac{x^3}{3} - \frac{4}{3}$$

$$y^{-1} = \frac{4 - x^3}{3}$$

$$y = \frac{3}{4 - x^3}$$

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$ and $y = 2$ when $x = 1$

$y = \sqrt{x^2 + 3}$, valid for all real numbers

2. $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and $y = 3$ when $x = 4$

$y = \sqrt{25 - x^2}$, valid on the interval $(-5, 5)$

$$y \cdot \frac{dy}{dx} = \frac{x}{y} \cdot dx$$

$$\int y\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$\frac{1}{2}(4) = \frac{1}{2} + C$$

$$2 = \frac{1}{2} + C$$

$$C = 3/2$$

$$\left(\frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{3}{2}\right)\frac{2}{1}$$

$$\sqrt{y^2} = \sqrt{x^2 + 3}$$

$$y = \sqrt{x^2 + 3}$$

In Exercises 1–10, use separation of variables to solve the initial value problem. Indicate the domain over which the solution is valid.

1. $\frac{dy}{dx} = \frac{x}{y}$ and $y = 2$ when $x = 1$
   $y = \sqrt{x^2 + 3}$, valid for all real numbers
2. $\frac{dy}{dx} = -\frac{x}{y}$ and $y = 3$ when $x = 4$
   $y = \sqrt{25 - x^2}$, valid on the interval $(-5, 5)$
3. $\frac{dy}{dx} = \frac{y}{x}$ and $y = 2$ when $x = 2$
   $y = x$, valid on the interval $(0, \infty)$
4. $\frac{dy}{dx} = 2xy$ and $y = 3$ when $x = 0$
   $y = 3e^{x^2}$, valid for all real numbers
5. $\frac{dy}{dx} = (y + 5)(x + 2)$ and $y = 1$ when $x = 0$
   $y = 6e^{x^2/2+2x} - 5$, valid for all real numbers

$y+5$

$e \approx 2.71$

If $a = b$, then $e^a = e^b$

(5) $\int \frac{1}{y+5}\, dy = \int (x+2)\, dx$

$\ln(y+5) = \frac{x^2}{2} + 2x + C$ $\quad y=1,\ x=0$

$\ln 6 = C$

$\ln(y+5) = \left(\frac{x^2}{2} + 2x + \ln 6\right)$

$e \qquad e^{\left(\frac{x^2}{2} + 2x + \ln 6\right)}$

$e^a \cdot e^b = e^{a+b}$

$y+5 = e^{\left(\frac{x^2}{2} + 2x + \ln 6\right)}$

6. $\dfrac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$

$y = \tan^{-1} x$, valid for all real numbers

$$\frac{1}{\cos\theta} = \sec\theta$$

$$\frac{dx \cdot}{\cos^2 y}\frac{dy}{dx} = \frac{\cos^2 y}{\cos^2 y} \cdot dx$$

$$\int \sec^2 y\, dy = \int 1\, dx$$

$$\tan y = x$$

... then use inverse trig!

$y = (x^2 + 3)^{-1}$, valid for all real numbers

10. $\dfrac{dy}{dx} = \dfrac{4\sqrt{y}\ln x}{x}$ and $y = 1$ when $x = e$

$y = (\ln x)^4$, valid on the interval $(0, \infty)$

$$\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}$$

$u = \ln x$

$du = \frac{1}{x}$

$$\frac{1}{4\sqrt{y}}\,dy = \frac{\ln x}{x}\,dx$$

$$\frac{1}{4}\int y^{-1/2} = \int u\,du$$

$$\frac{1}{4}\left(\frac{-2}{3}y^{-3/2}\right) = u^2$$

$$-\frac{1}{6}y^{-3/2} = (\ln x)^2 + C$$

$$-\frac{1}{6} = 1 + C$$

$$-\frac{1}{6}y^{-3/2} = (\ln x)^2 - \frac{1}{6}$$

In Exercises 1–10, use separation of variables to solve the initial value problem. Indicate the domain over which the solution is valid.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$ and $y = 2$ when $x = 1$
$y = \sqrt{x^2 + 3}$, valid for all real numbers

2. $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and $y = 3$ when $x = 4$
$y = \sqrt{25 - x^2}$, valid on the interval $(-5, 5)$

3. $\dfrac{dy}{dx} = \dfrac{y}{x}$ and $y = 2$ when $x = 2$
$y = x$, valid on the interval $(0, \infty)$

4. $\dfrac{dy}{dx} = 2xy$ and $y = 3$ when $x = 0$
$y = 3e^{x^2}$, valid for all real numbers

5. $\dfrac{dy}{dx} = (y + 5)(x + 2)$ and $y = 1$ when $x = 0$
$y = 6e^{x^2/2+2x} - 5$, valid for all real numbers

6. $\dfrac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$
$y = \tan^{-1} x$, valid for all real numbers

7. $\dfrac{dy}{dx} = (\cos x)e^{y+\sin x}$ and $y = 0$ when $x = 0$
$y = -\ln(2 - e^{\sin x})$, valid for all real numbers

8. $\dfrac{dy}{dx} = e^{x-y}$ and $y = 2$ when $x = 0$
$y = \ln(e^x + e^2 - 1)$, valid for all real numbers

9. $\dfrac{dy}{dx} = -2xy^2$ and $y = 0.25$ when $x = 1$
$y = (x^2 + 3)^{-1}$, valid for all real numbers

10. $\dfrac{dy}{dx} = \dfrac{4\sqrt{y}\ln x}{x}$ and $y = 1$ when $x = e$
$y = (\ln x)^4$, valid on the interval $(0, \infty)$

$x^a x^b = x^{a+b}$

7 $(\cos x)e^{y+\sin x}$

8 e^{x-y}

$\dfrac{dy}{dx} = \dfrac{e^x}{e^y}$

$\int e^y\,dy = \int e^x\,dx$

$e^y = e^x + C$

$e^2 = e^0 + C$

$e^2 = 1 + C$

$e^2 - 1 = C$

$\ln(e^y) = \ln(e^x + e^2 - 1)$

$y = \ln(e^x + e^2 - 1)$

9. $\dfrac{dy}{dx} = -2xy^2$ and $y = 0.25$ when $x = 1$

$y = (x^2 + 3)^{-1}$, valid for all real numbers

10. $\dfrac{dy}{dx} = \dfrac{4\sqrt{y}\ln x}{x}$ and $y = 1$ when $x = e$

$y = (\ln x)^4$, valid on the interval $(0, \infty)$

9 $\dfrac{dy}{dx} = -2xy^2$

$\displaystyle\int \frac{1}{y^2}\,dy = \int -2x\,dx$

$\displaystyle\int y^{-2}\,dy = \int -2x\,dx$

$-y^{-1} = -x^2 + C$

$-\dfrac{1}{.25} = -1^2 + C$

$-4 = -1 + C$

$C = -3$

$\dfrac{+1}{y} = +x^2 + 3$

$\dfrac{1}{1}$

$y = \dfrac{1}{(x^2+3)}$

7. $\dfrac{dy}{dx} = (\cos x)e^{y+\sin x}$ and $y = 0$ when $x = 0$

$y = -\ln(2 - e^{\sin x})$, valid for all real numbers

$$\frac{dy}{dx} = (\cos x)\, e^{y+\sin x}$$

$$\frac{dy}{dx} = (\cos x)(e^{y})e^{\sin x}$$

$$\int \frac{1}{e^{y}}\,dy = \int e^{\sin x}\cdot \cos x\,dx$$

$$e^{u}\cdot u'$$

$u = \sin x$

$u' = \cos x$

$$\frac{d}{dx}e^{u} = e^{u}\cdot u'$$

$$\int e^{-y}\,dy = e^{\sin x} + C$$

$$-e^{-y}$$

9. $\dfrac{dy}{dx} = -2xy^2$ and $y = 0.25$ when $x = 1$

$y = (x^2 + 3)^{-1}$, valid for all real numbers

10. $\dfrac{dy}{dx} = \dfrac{4\sqrt{y}\ln x}{x}$ and $y = 1$ when $x = e$

$y = (\ln x)^4$, valid on the interval $(0, \infty)$

10

$u = \ln x$

$du = \frac{1}{x}dx$

$\frac{1}{4y^{\frac{1}{2}}}dy = \frac{1}{x}\cdot \ln x \, dx$

$\frac{1}{4}\int y^{-\frac{1}{2}}dy = \int u \, du$

$\frac{1}{4}\cdot\frac{2}{1}y^{\frac{1}{2}}$

$\frac{1}{2}y^{\frac{1}{2}} = \frac{1}{2}u^2 + C$

$\frac{1}{2}y^{\frac{1}{2}} = \frac{1}{2}(\ln x)^2 + C$

$\frac{1}{2} = \frac{1}{2} + C$

$C = 0$

$2\left(\frac{1}{2}y^{\frac{1}{2}}\right) = 2\left(\frac{1}{2}(\ln x)^2\right)$

$\left(y^{\frac{1}{2}}\right)^2 = \left((\ln x)^2\right)^2$

"Where does all this precalculus stuff fit in calculus?"

[exponential]

The differential equation that describes this growth is $dy/dt = ky$, where k is the *growth constant* (if positive) or the *decay constant* (if negative). We can solve this equation by separating the variables.

$x^{a+b} = x^a \cdot x^b$

continuous compound interest formula

$A = Pe^{rt}$

$$\frac{dy}{dt} = ky$$

$$\int \frac{1}{y}\,dy = \int k\,dt \qquad \text{Separate the variables}$$

$$\ln|y| = kt + C \qquad \text{Antidifferentiate both sides}$$

$e \qquad e$

$$|y| = e^{kt+C} \qquad \text{Exponentiate both sides}$$

$$|y| = e^C e^{kt} \qquad \text{Property of exponents}$$

$$y = \pm e^C e^{kt} \qquad \text{Definition of absolute value}$$

$$y = Ae^{kt} \qquad \text{Let } A = \pm e^C.$$

The Law of Exponential Change

If y changes at a rate proportional to the amount present (that is, if $dy/dt = ky$), and if $y = y_0$ when $t = 0$, then

$$y = y_0 e^{kt}.$$

The constant k is the **growth constant** if $k > 0$ or the **decay constant** if $k < 0$.

Now try Exercise 11.

In Exercises 11–14, find the solution of the differential equation $dy/dt = ky$, k a constant, that satisfies the given conditions.

11. $k = 1.5$, $y(0) = 100$
$y(t) = 100e^{1.5t}$

12. $k = -0.5$, $y(0) = 200$
$y(t) = 200e^{-0.5t}$

13. $y(0) = 50$, $y(5) = 100$
$y(t) = 50e^{(0.2 \ln 2)t}$

14. $y(0) = 60$, $y(10) = 30$
$y(t) = 60e^{-(0.1 \ln 2)t}$

11

$y = y_0 e^{kt}$

$y(0) = 100$

$t = 0$

$y = 100$

$100 = y_0 e^{0}$

$100 = y_0$

$y = 100e^{1.5t}$

In Exercises 11–14, find the solution of the differential equation $dy/dt = ky$, k a constant, that satisfies the given conditions.

11. $k = 1.5, \quad y(0) = 100$
$y(t) = 100e^{1.5t}$

12. $k = -0.5, \quad y(0) = 200$
$y(t) = 200e^{-0.5t}$

13. $y(0) = 50, \quad y(5) = 100$
$y(t) = 50e^{(0.2 \ln 2)t}$

14. $y(0) = 60, \quad y(10) = 30$
$y(t) = 60e^{-(0.1 \ln 2)t}$

13

$y = y_0 e^{kt}$

$t = 0$
$y = 50$

$t = 5$
$y = 100$

$50 = y_0 e^{k(0)}$

$50 = y_0 e^{0}$

$50 = y_0$

$y = 50e^{kt}$

$\frac{100}{50} = \frac{50e^{5k}}{50}$

$2 = e^{5k}$

$\frac{\ln 2}{5} = \frac{5k}{5}$

$k = \frac{\ln 2}{5} = .2\ln 2$

$y(t) = 50e^{.2\ln 2 t}$

In Exercises 15–18, complete the table for an investment if interest is compounded continuously.

	Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
15.	1000	8.6		
16.	2000	k	15	
17.		5.25		2898.44
18.	1200			10,405.37

$2y_0$

15. 8.06 yr doubling time; $13,197.14 in 30 yr **16.** 4.62% rate; $8000 in 30 yr

17. $600 initially; 13.2 yr doubling time **18.** 7.2% rate; 9.63 yr doubling time

16

$$y = y_0 e^{kt}$$

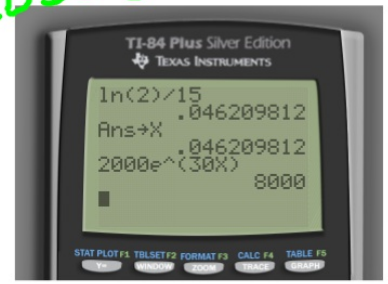
$$\frac{2\cancel{y_0}}{\cancel{y_0}} = \frac{\cancel{y_0} e^{k(15)}}{\cancel{y_0}}$$

$$2 = e^{k(15)}$$

$$\frac{\ln 2}{15} = \frac{15k}{15}$$

$$k = \frac{\ln 2}{15}$$

$$y = 2000 e^{k(30)}$$



EXAMPLE 2 Compounding Interest Continuously

Suppose you deposit \$800 in an account that pays 6.3% annual interest. How much will you have 8 years later if the interest is **(a)** compounded continuously? **(b)** compounded quarterly?

$A = A_0\left(1+\frac{k}{n}\right)^{kt}$

$k = r \ldots$

SOLUTION

Here $A_0 = 800$ and $r = 0.063$. The amount in the account to the nearest cent after 8 years is

(a) $A(8) = 800e^{(0.063)(8)} = 1324.26.$

(b) $A(8) = 800\left(1 + \frac{0.063}{4}\right)^{(4)(8)} = 1319.07.$

"quarterly"

$n = 4$

You might have expected to generate more than an additional \$5.19 with interest compounded continuously. *Now try Exercise 19.*

In Exercises 19 and 20, find the amount of time required for a $2000 investment to double if the annual interest rate r is compounded **(a)** annually, **(b)** monthly, **(c)** quarterly, and **(d)** continuously.

$n = 1$ $n = 12$ $n = 4$

e

19. $r = 4.75\%$

20. $r = 8.25\%$

(a) 8.74 yr **(b)** 8.43 yr **(c)** 8.49 yr **(d)** 8.40 yr

EXAMPLE 3 Finding Half-Life

Find the half-life of a radioactive substance with decay equation $y = y_0 e^{-kt}$ and show that the half-life depends only on k.

SOLUTION

Model The half-life is the solution to the equation

$$y_0 e^{-kt} = \frac{1}{2} y_0.$$

continued

Solve Algebraically

$$e^{-kt} = \frac{1}{2} \qquad \text{Divide by } y_0.$$

$$-kt = \ln \frac{1}{2} \qquad \text{Take ln of both sides.}$$

$$t = -\frac{1}{k} \ln \frac{1}{2} = \frac{\ln 2}{k} \qquad \ln \frac{1}{a} = -\ln a$$

Interpret This value of t is the half-life of the element. It depends only on the value of k. Note that the number y_0 does not appear. ***Now try Exercise 21.***

EXAMPLE 4 Choosing a Base

At the beginning of the summer, the population of a hive of bald-faced hornets (which are actually wasps) is growing at a rate proportional to the population. From a population of 10 on May 1, the number of hornets grows to 50 in thirty days. If the growth continues to follow the same model, how many days after May 1 will the population reach 100?

SOLUTION

Since $dy/dt = ky$, the growth is exponential. Noting that the population grows by a factor of 5 in 30 days, we model the growth in base 5: $y = 10 \times 5^{(1/30)t}$. Now we need only solve the equation $100 = 10 \times 5^{(1/30)t}$ for t:

$$100 = 10 \times 5^{(1/30)t}$$

$$10 = 5^{(1/30)t}$$

$$\ln 10 = (1/30)t \ln 5$$

$$t = 30\left(\frac{\ln 10}{\ln 5}\right) = 42.920$$

Approximately 43 days will pass after May 1 before the population reaches 100.

Now try Exercise 23.

21. *Half-Life* The radioactive decay of Sm-151 (an isotope of samarium) can be modeled by the differential equation $dy/dt = -0.0077y$, where t is measured in years. Find the half-life of Sm-151. 90 years.

22. *Half-Life* An isotope of neptunium (Np-240) has a half-life of 65 minutes. If the decay of Np-240 is modeled by the differential equation $dy/dt = -ky$, where t is measured in minutes, what is the decay constant k? $k = .01067$

23. *Growth of Cholera Bacteria* Suppose that the cholera bacteria in a colony grows unchecked according to the Law of Exponential Change. The colony starts with 1 bacterium and doubles in number every half hour.

(a) How many bacteria will the colony contain at the end of 24 h? 2.8×10^{14} bacteria

←look @ example 4 first!

22

$t = 65$

$\frac{1}{2}A_0 = A_0 e^{k(65)}$

$\frac{1}{2} = e^{65k}$

$\ln(.5) = 65k$

$\frac{\ln(.5)}{65} = \frac{65k}{65}$