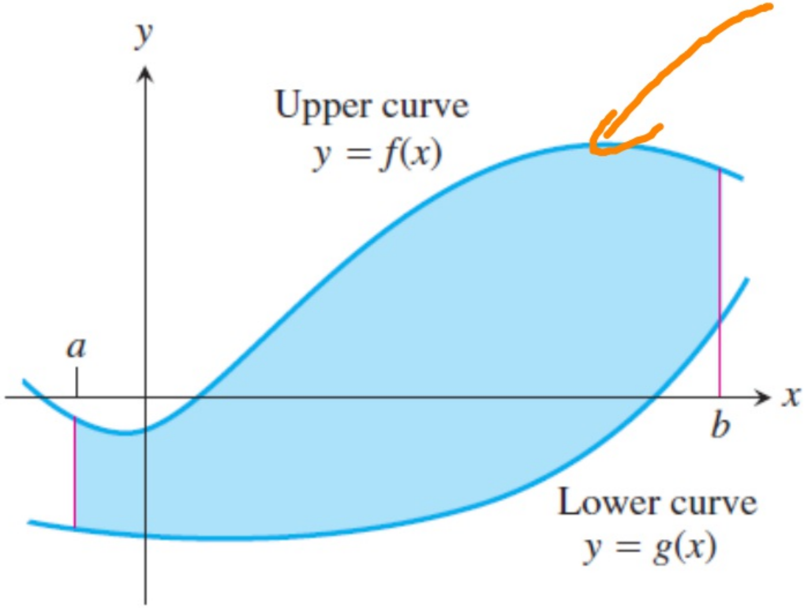
What you'll learn about

- Area Between Curves
- Area Enclosed by Intersecting Curves
- Boundaries with Changing Functions
- Integrating with Respect to *y*

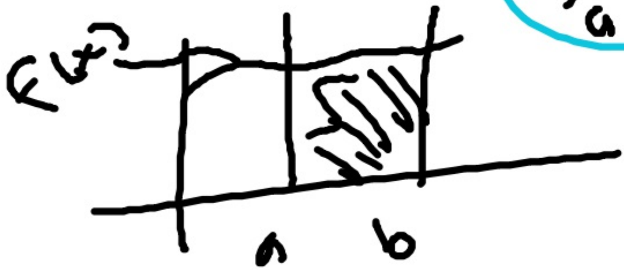
7.2 Areas in the Plane

We know how to find areas between a graph and the x axis, but what if we wanted to find the area between curves?

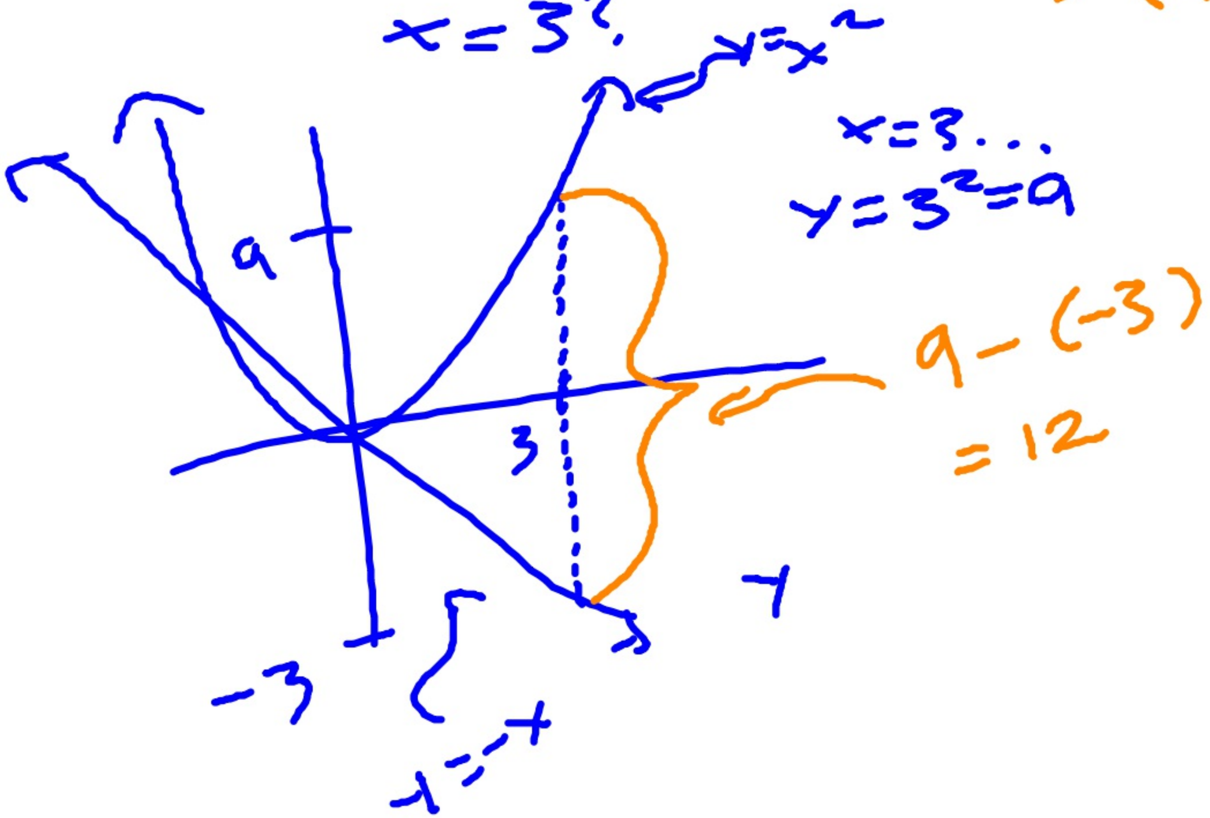


Before . . . .

$$\int_a^b f(x)$$



Q: What is the distance between $y = x^2$ and $y = -x$ when $x = 3$?



Since we are finding a distance between two points along a number line (y-axis), we can subtract the two values to express the total distance.

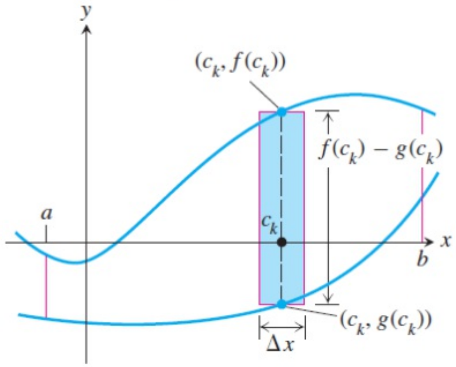


Figure 7.5 The area of a typical rectangle is $[f(c_k) - g(c_k)]\,\Delta x$.

$$[f(c_k) - g(c_k)]\,\Delta x$$

(typical area of one rectangle)

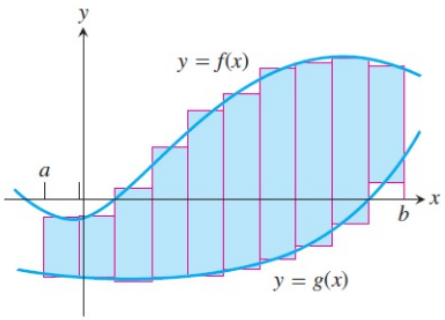


Figure 7.4 We approximate the region with rectangles perpendicular to the x-axis.

$$\sum [f(c_k) - g(c_k)]\,\Delta x.$$

(typical sum of the area of rectangles, or region)

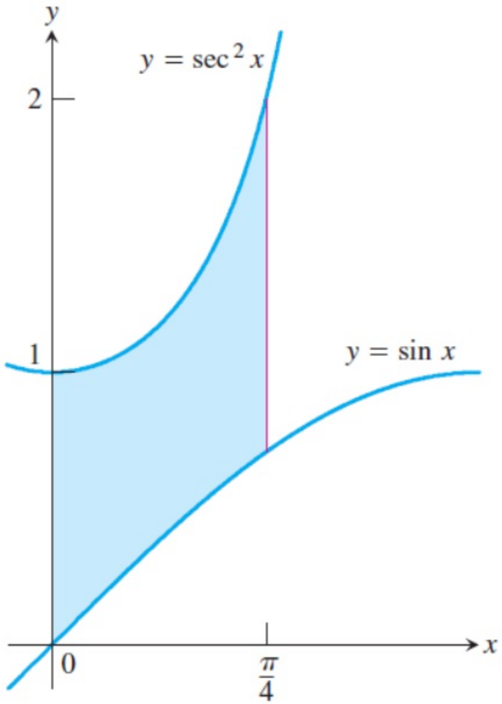
The limit of these sums as $\Delta x \to 0$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

DEFINITION Area Between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $[f - g]$ from a to b,

$$A = \int_a^b [f(x) - g(x)]\, dx.$$



EXAMPLE 1 Applying the Definition

Find the area of the region between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.

SOLUTION

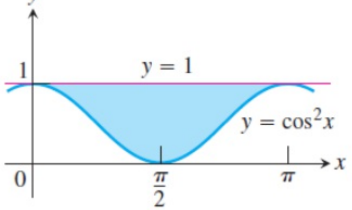
We graph the curves (Figure 7.6) to find their relative positions in the plane, and see that $y = \sec^2 x$ lies *above* $y = \sin x$ on $[0, \pi/4]$. The area is therefore

$$\begin{aligned} A &= \int_0^{\pi/4} [\sec^2 x - \sin x]\, dx \\ &= \Big[\tan x + \cos x\Big]_0^{\pi/4} \\ &= \frac{\sqrt{2}}{2} \text{ units squared.} \end{aligned}$$

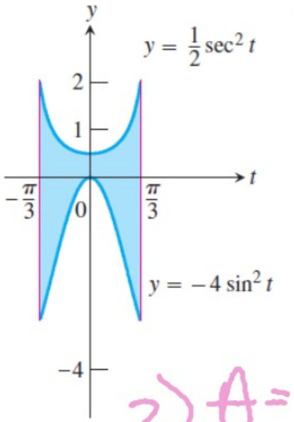
Now try Exercise 1.

In Exercises 1–6, find the area of the shaded region analytically.

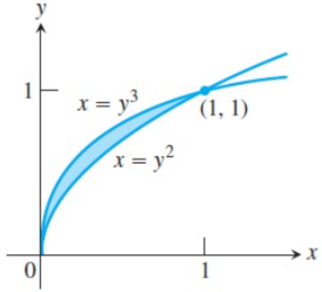
1. $\pi/2$



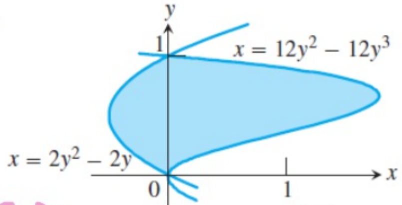
2. $4\pi/3$



3. $1/12$



4. $4/3$



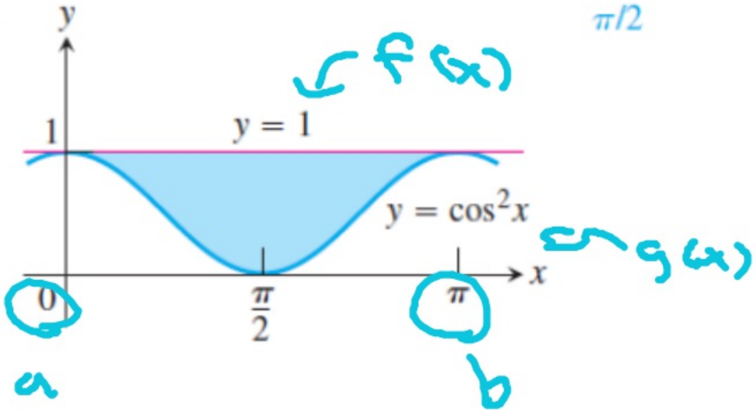
2) $A = \int_{-\pi/3}^{\pi/3} \left[\left(\frac{1}{2}\sec^2 t\right) - \left(-4\sin^2 t\right)\right] dt$

$= 4.18$

$A = \int_0^1 \left[\left(12y^2 - 12y^3\right) - \left(2y^2 - 2y\right)\right] dy$

$= 1.\overline{33}$

1.

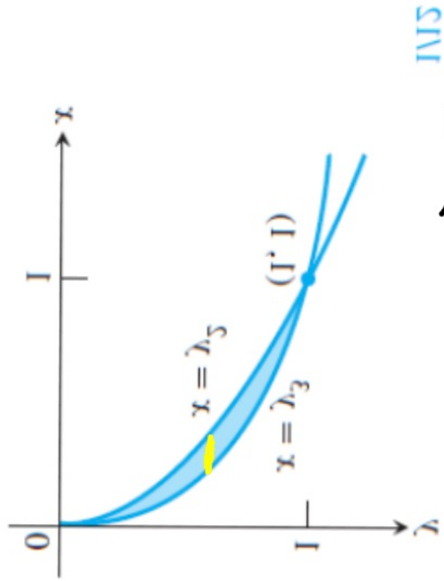


$\pi/2$

```
∫_0^π (1-(cos(X))²)d
        1.570796327
π/2
        1.570796327
```

$$A = \int_0^{\pi} [1 - \cos^2 x]\, dx$$

$\frac{1}{12} = .08\overline{33}$



$$\int_0^1 (y^2 - y^3)\, dy$$

$\int_0^1 (X^2 - X^3)\,dX$

.0833333333

OR....

$$\sqrt[3]{x} = \sqrt[3]{y^3}$$

$$\cancel{y} = \sqrt[3]{x}$$

3.

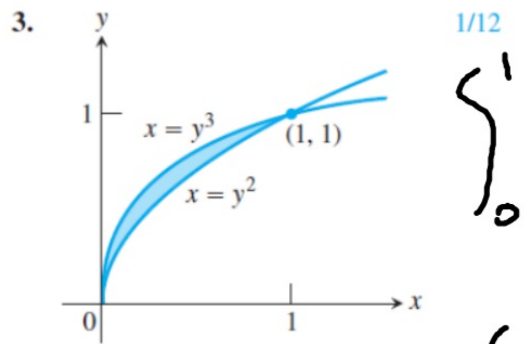
???

$$\int_0^1 y^3 - y^2\, dx \quad \sqrt{y^2} = \sqrt{x}$$

$$y = \sqrt{x}$$

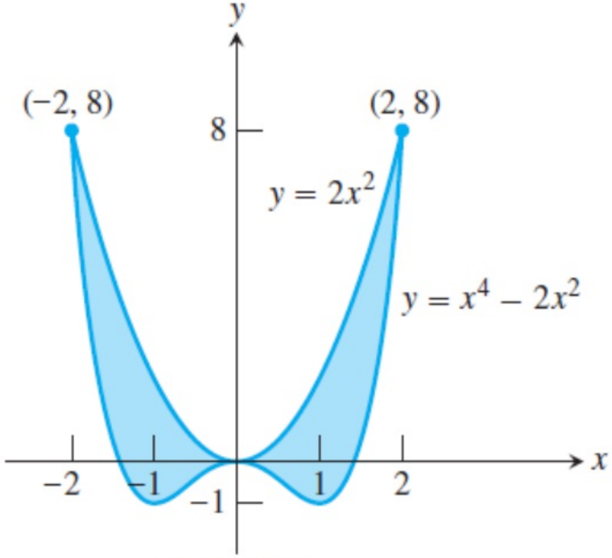
$\int_0^1 (\sqrt[3]{X} - \sqrt{X})\,dX$

.0833340645



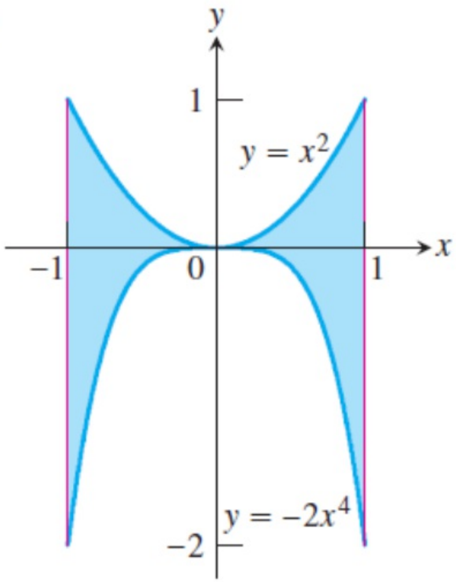
$$\int_0^1 \left(\sqrt[3]{x} - \sqrt{x}\right) dx$$

5. 128/15



NOT TO SCALE

6. 22/15



I don't have to break down the integration because one graph is *always* above the other graph.

EXAMPLE 2 Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

SOLUTION

We graph the curves to view the region (Figure 7.8).

The limits of integration are found by solving the equation

$$2 - x^2 = -x$$

either algebraically or by calculator. The solutions are $x = -1$ and $x = 2$.

continued

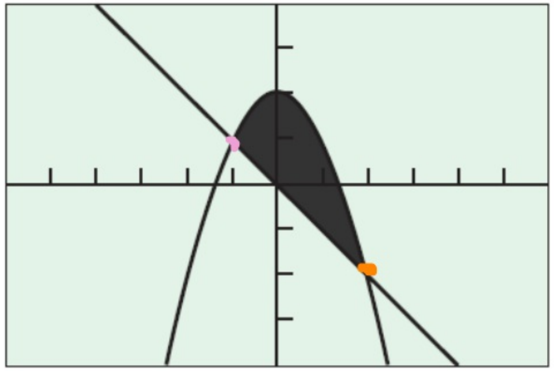
Since the parabola lies above the line on $[-1, 2]$, the area integrand is $2 - x^2 - (-x)$.

$$A = \int_{-1}^{2} [2 - x^2 - (-x)]\, dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \frac{9}{2} \text{ units squared}$$

$y_1 = 2 - x^2$
$y_2 = -x$



[–6, 6] by [–4, 4]

Figure 7.8 The region in Example 2.

EXAMPLE 3 Using a Calculator

Find the area of the region enclosed by the graphs of $y = 2 \cos x$ and $y = x^2 - 1$.

SOLUTION

The region is shown in Figure 7.9.

Using a calculator, we solve the equation

$$2 \cos x = x^2 - 1$$

to find the x-coordinates of the points where the curves intersect. These are the limits of integration. The solutions are $x = \pm 1.265423706$. We store the negative value as A and the positive value as B. The area is

$$\text{NINT}\ (2 \cos x - (x^2 - 1), x, A, B) \approx 4.994907788.$$

This is the final calculation, so we are now free to round. The area is about 4.99.

Now try Exercise 7.

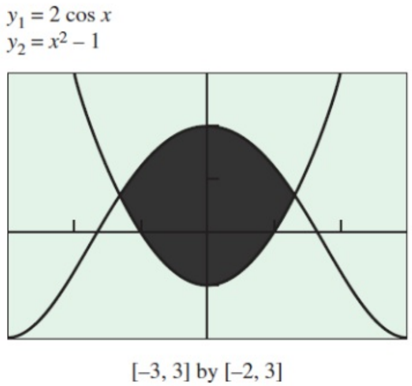


Figure 7.9 The region in Example 3.

In Exercises 7 and 8, use a calculator to find the area of the region enclosed by the graphs of the two functions.

7. $y = \sin x, y = 1 - x^2$ ≈ 1.670 **8.** $y = \cos(2x), y = x^2 - 2$ ≈ 4.332