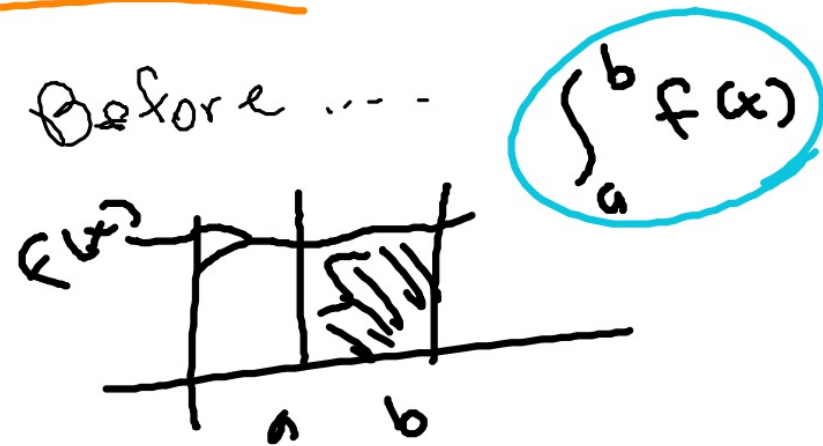
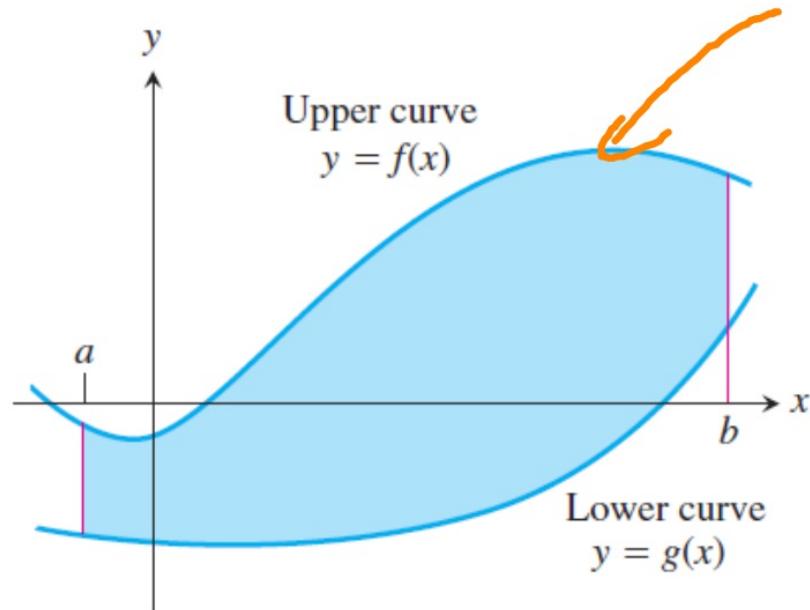


What you'll learn about

- Area Between Curves
- Area Enclosed by Intersecting Curves
- Boundaries with Changing Functions
- Integrating with Respect to y

7.2 Areas in the Plane

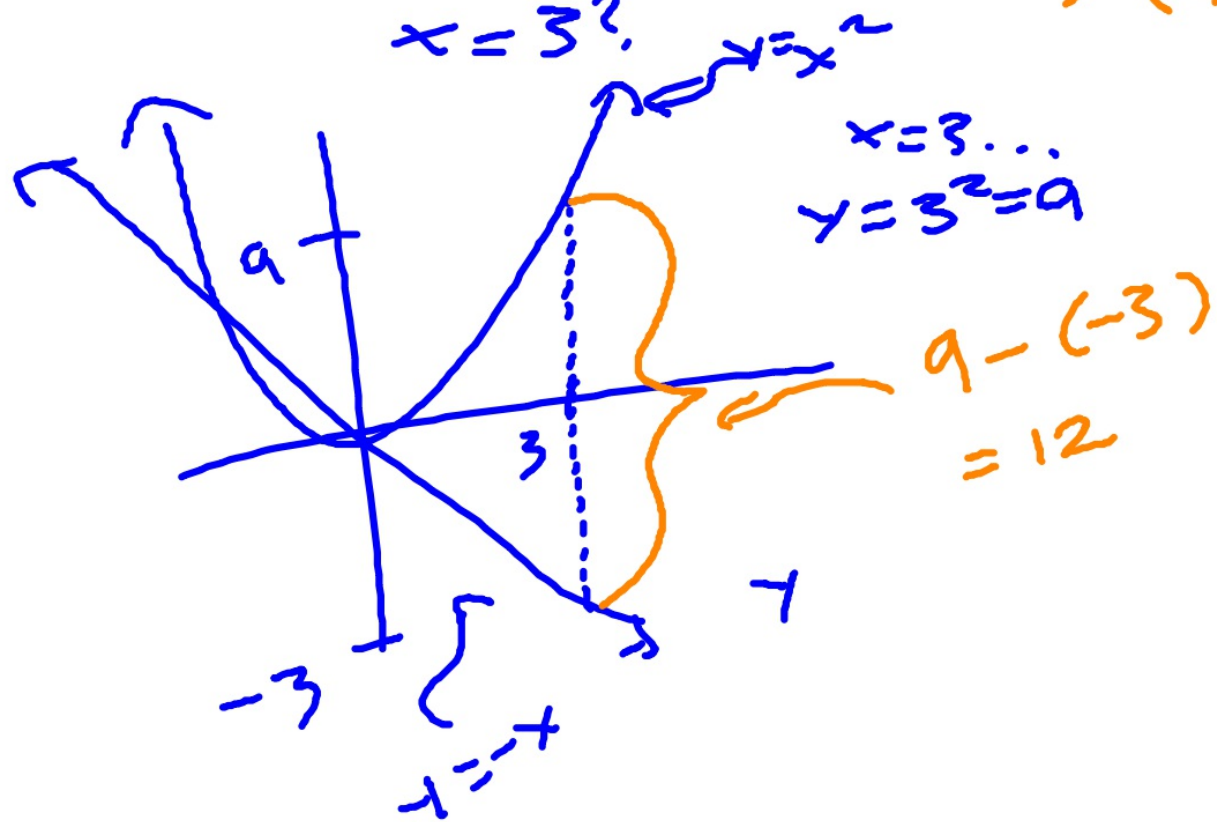
We know how to find areas between a graph and the x axis, but what if we wanted to find the area between curves?



Q: What is the distance
between $y = x^2$ and
 $y = -x$ when

$x = 3$?

3 (-9)



Since we are finding a distance between two points along a number line (y-axis), we can subtract the two values to express the total distance.

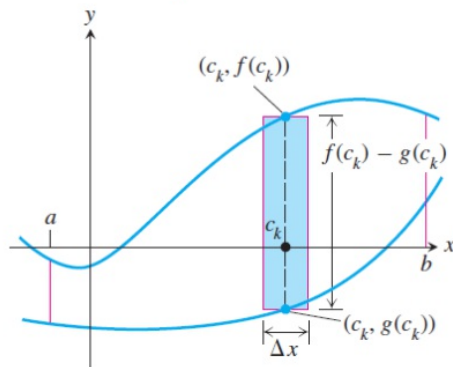


Figure 7.5 The area of a typical rectangle is $[f(c_k) - g(c_k)] \Delta x$.

$$[f(c_k) - g(c_k)] \Delta x$$

(typical area of one rectangle)

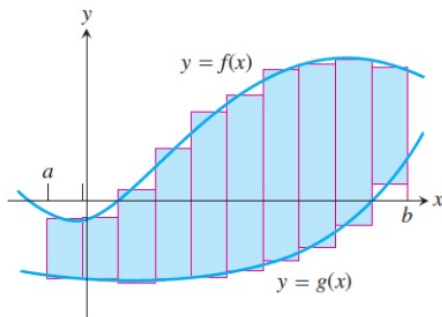


Figure 7.4 We approximate the region with rectangles perpendicular to the x-axis.

$$\sum [f(c_k) - g(c_k)] \Delta x.$$

(typical sum of the area of rectangles, or region)

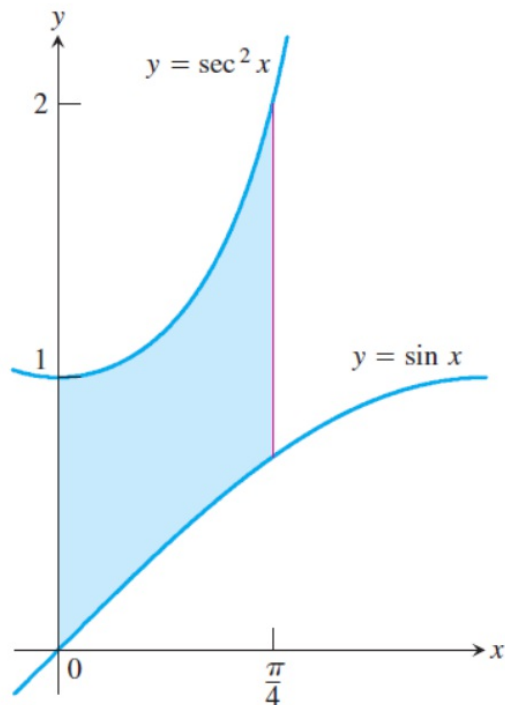
The limit of these sums as $\Delta x \rightarrow 0$ is

$$\int_a^b [f(x) - g(x)] dx.$$

DEFINITION Area Between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $[f - g]$ from a to b ,

$$A = \int_a^b [f(x) - g(x)] dx.$$



EXAMPLE 1 Applying the Definition

Find the area of the region between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.

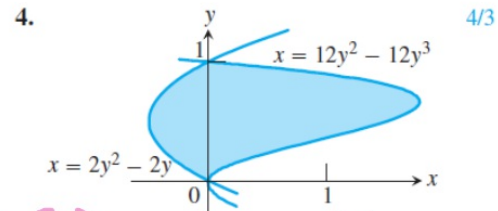
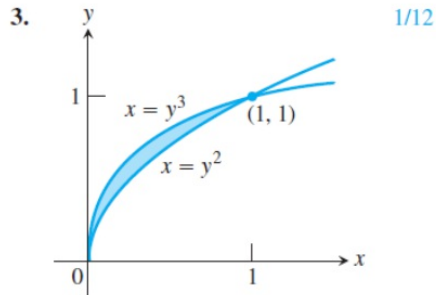
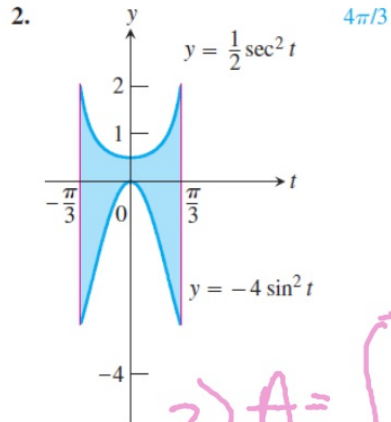
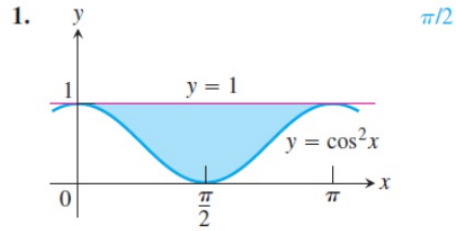
SOLUTION

We graph the curves (Figure 7.6) to find their relative positions in the plane, and see that $y = \sec^2 x$ lies *above* $y = \sin x$ on $[0, \pi/4]$. The area is therefore

$$\begin{aligned} A &= \int_0^{\pi/4} [\sec^2 x - \sin x] dx \\ &= \left[\tan x + \cos x \right]_0^{\pi/4} \\ &= \frac{\sqrt{2}}{2} \text{ units squared.} \end{aligned}$$

Now try Exercise 1.

In Exercises 1–6, find the area of the shaded region analytically.



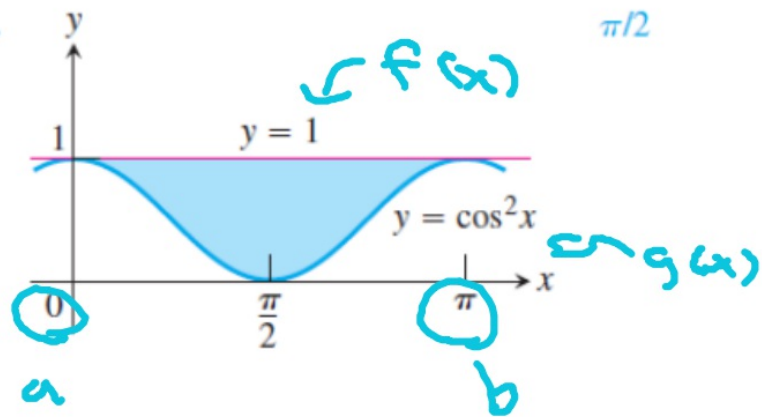
$$2) A = \int_{-\pi/3}^{\pi/3} \left[\left(\frac{1}{2} \sec^2 t \right) - \left(-4 \sin^2 t \right) \right] dt$$

$$= 4.18$$

$$A = \int_0^1 \left[\left(12y^2 - 12y^3 \right) - \left(2y^2 - 2y \right) \right] dy$$

$$= 1.33$$

1.

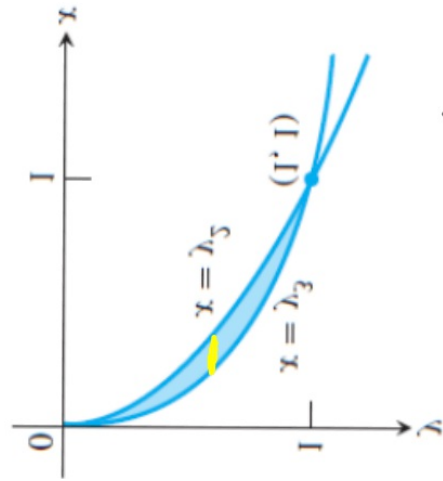


$$\int_0^{\pi} (1 - (\cos(x))^2) dx$$

	1.570796327
$\pi/2$	1.570796327

$$A = \int_0^{\pi} [1 - \cos^2 x] dx$$

$$\frac{1}{12} = .0833$$



1/15

$$\int_0^1 (y^2 - y^3) dy$$

$$\int_0^1 (x^2 - x^3) dx$$

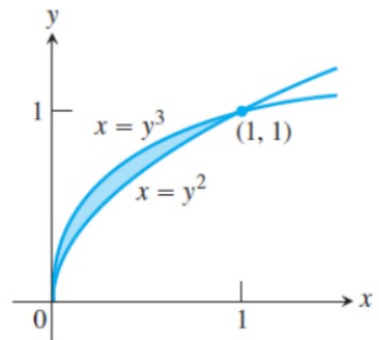
.0833333333

OR...

$$\sqrt[3]{x} = \sqrt[3]{y^3}$$

$$y = \sqrt[3]{x}$$

3.



1/12

$$\int_0^1 (y^{\frac{3}{2}} - y^{\frac{5}{2}}) dx$$

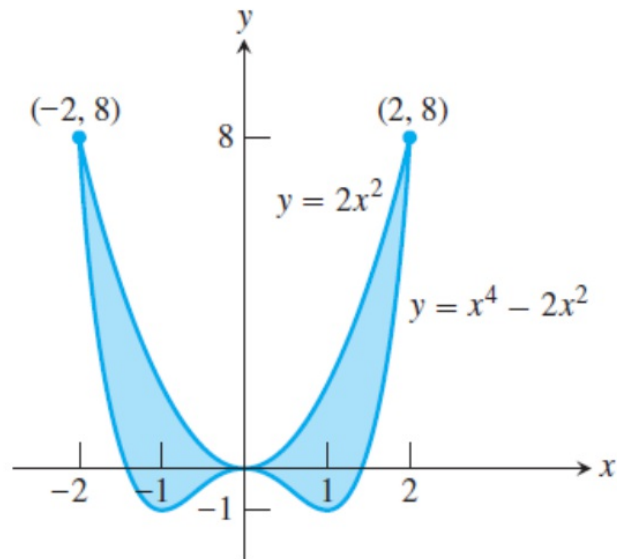
$$y = \sqrt{x}$$

$$\int_0^1 (\sqrt[3]{x} - \sqrt{x}) dx$$

.0833340645

$$\int_0^1 (\sqrt[3]{x} - \sqrt{x}) dx$$

5.



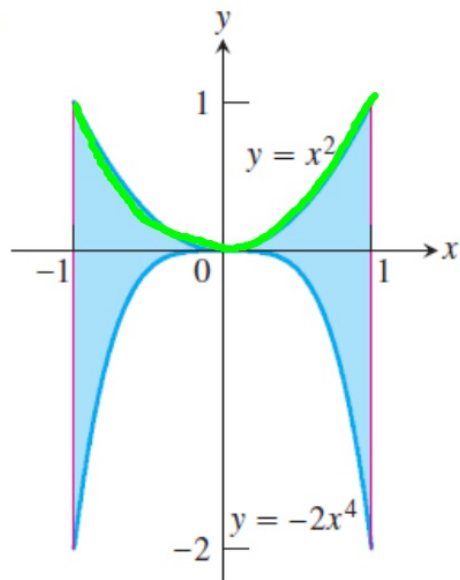
NOT TO SCALE

128/15

I don't have to break down the integration because one graph is *always* above the other graph.

$$\int_{-2}^2 [2x^2 - (x^4 - 2x^2)] dx$$

6.



22/15

$$\int_{-1}^1 [x^2 - (-2x^4)] dx$$

EXAMPLE 2 Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

SOLUTION

We graph the curves to view the region (Figure 7.8).

The limits of integration are found by solving the equation

$$2 - x^2 = -x$$

either algebraically or by calculator. The solutions are $x = -1$ and $x = 2$.

continued

Since the parabola lies above the line on $[-1, 2]$, the area integrand is $2 - x^2 - (-x)$.

$$\begin{aligned} A &= \int_{-1}^2 [2 - x^2 - (-x)] dx \\ &= \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^2 \\ &= \frac{9}{2} \text{ units squared} \end{aligned}$$

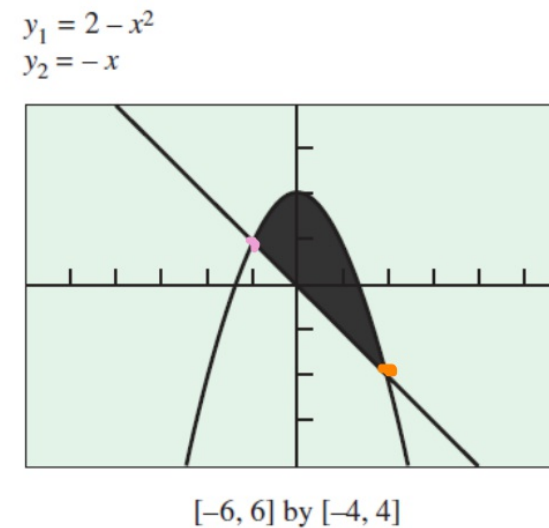


Figure 7.8 The region in Example 2.

EXAMPLE 3 Using a Calculator

Find the area of the region enclosed by the graphs of $y = 2 \cos x$ and $y = x^2 - 1$.

SOLUTION

The region is shown in Figure 7.9.

Using a calculator, we solve the equation

$$2 \cos x = x^2 - 1$$

to find the x -coordinates of the points where the curves intersect. These are the limits of integration. The solutions are $x = \pm 1.265423706$. We store the negative value as A and the positive value as B . The area is

$$\text{NINT}(2 \cos x - (x^2 - 1), x, A, B) \approx 4.994907788.$$

This is the final calculation, so we are now free to round. The area is about 4.99.

Now try Exercise 7.

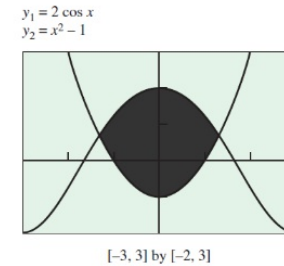
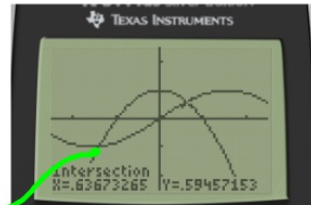
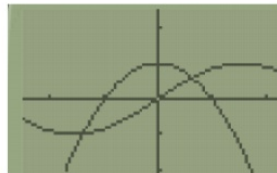
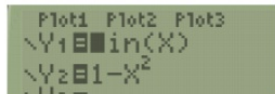


Figure 7.9 The region in Example 3.

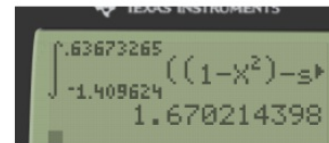
In Exercises 7 and 8, use a calculator to find the area of the region enclosed by the graphs of the two functions.

7. $y = \sin x, y = 1 - x^2 \approx 1.670$ 8. $y = \cos(2x), y = x^2 - 2 \approx 4.332$



$$\int_{-1.409624}^{.63673265} [(1-x^2) - \sin(x)] dx$$

1.409...

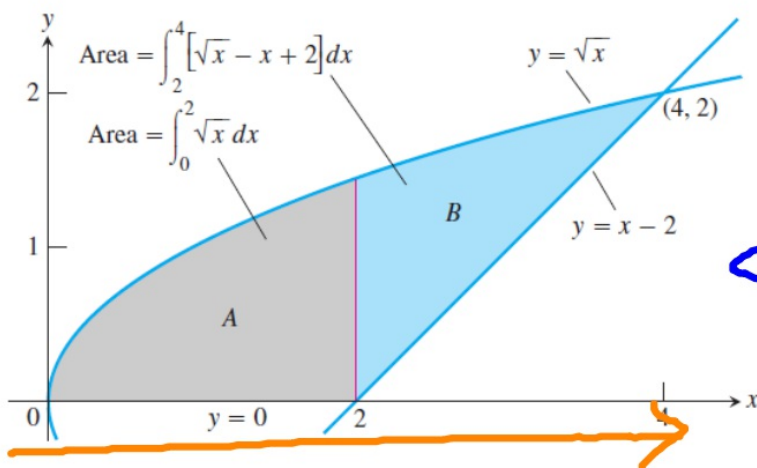


EXAMPLE 4 Finding Area Using Subregions

Find the area of the region R in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x -axis and the line $y = x - 2$.

SOLUTION

The region is shown in Figure 7.10.



$$\int_a^b f(x) dx$$

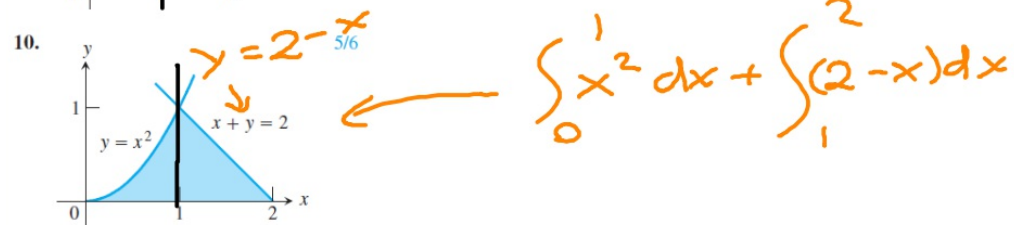
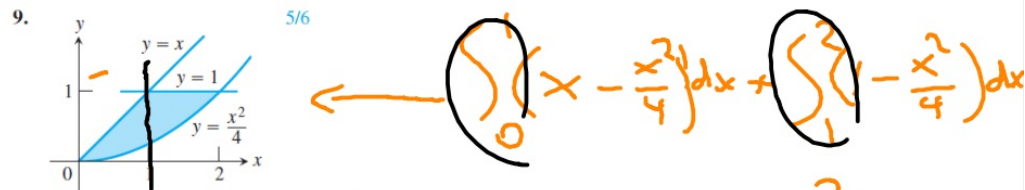
$$\text{Area of } R = \underbrace{\int_0^2 \sqrt{x} dx}_{\text{area of } A} + \underbrace{\int_2^4 [\sqrt{x} - (x - 2)] dx}_{\text{area of } B}$$

$$= \left. \frac{2}{3} x^{3/2} \right|_0^2 + \left. \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right] \right|_2^4$$

$$= \frac{10}{3} \text{ units squared}$$

Now try Exercise 9.

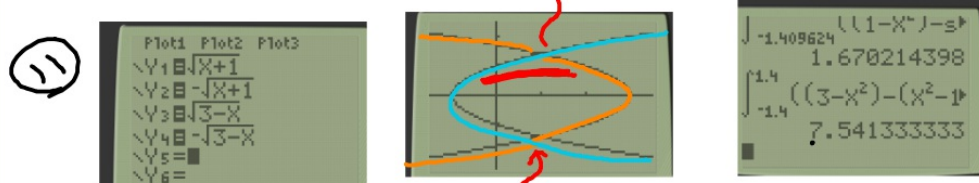
In Exercises 9 and 10, find the area of the shaded region analytically.



In Exercises 11 and 12, find the area enclosed by the graphs of the two curves by integrating with respect to y.

We will do this after going through example 5...

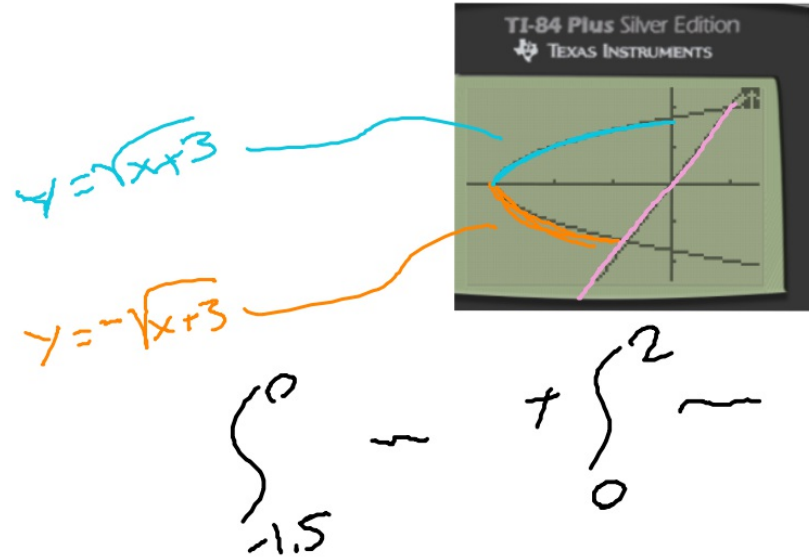
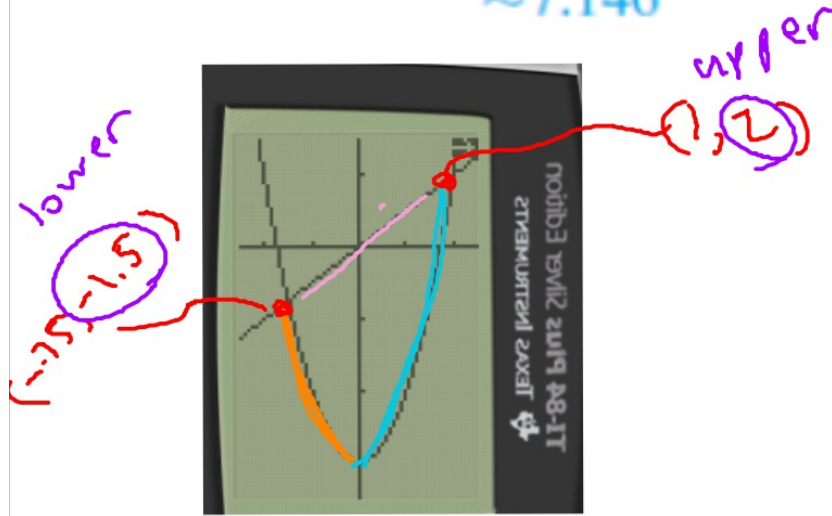
11. $y^2 = x + 1, y^2 = 3 - x \approx 7.542$ 12. $y^2 = x + 3, y = 2x \approx 7.146$



$$\begin{cases} y^2 = 3 - x \\ x = 3 - y^2 \end{cases} \quad \begin{cases} y^2 = x + 1 \\ x = y^2 + 1 \end{cases}$$

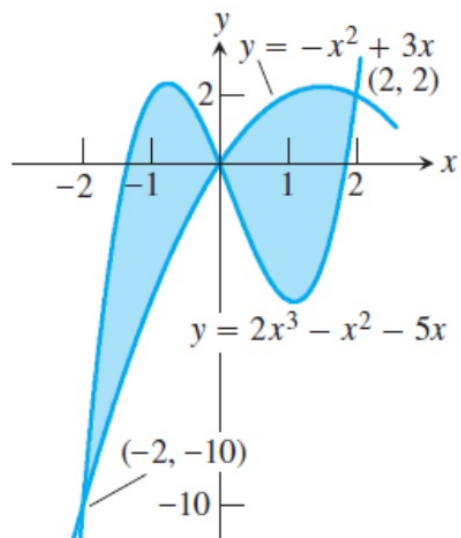
$$\int_{-1.4}^{1.4} [(3 - y^2) - (y^2 + 1)] dy$$

12. $y^2 = x + 3, y = 2x$
 ≈ 7.146



In Exercises 13 and 14, find the total shaded area.

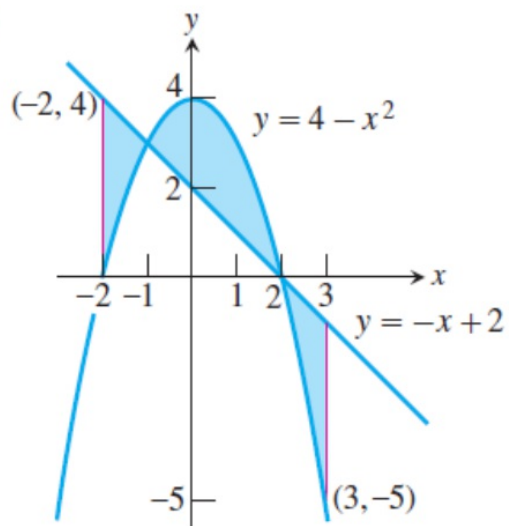
13.



16

$$\int_{-2}^0 (2x^3 - x^2 - 5x) - (-x^2 + 3x) dx$$

14.



$8\frac{1}{6}$

$$+ \int_0^2 (-x^2 + 3x) - (2x^3 - x^2 - 5x) dx$$