What you'll learn about

- Volume As an Integral
- Square Cross Sections
- Circular Cross Sections
- Cylindrical Shells
- Other Cross Sections

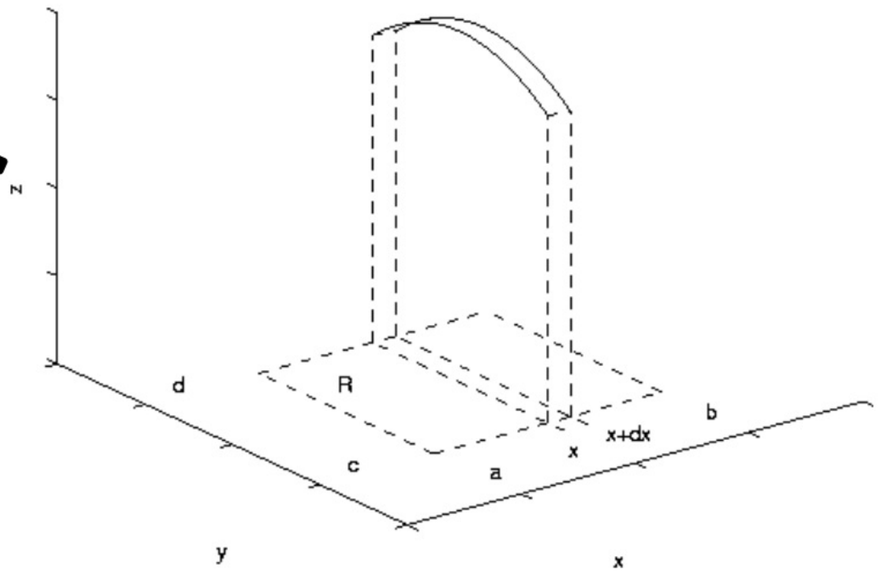
. . . and why

The techniques of this section allow us to compute volumes of certain solids in three dimensions.

7.3 Volumes

In this section, we are going to generate volumes with known cross sections.

First, let's think of it like a loaf of sliced bread. We will find the volume of each "slice," and then add them all up as the thickness approaches 0 (integrate!)



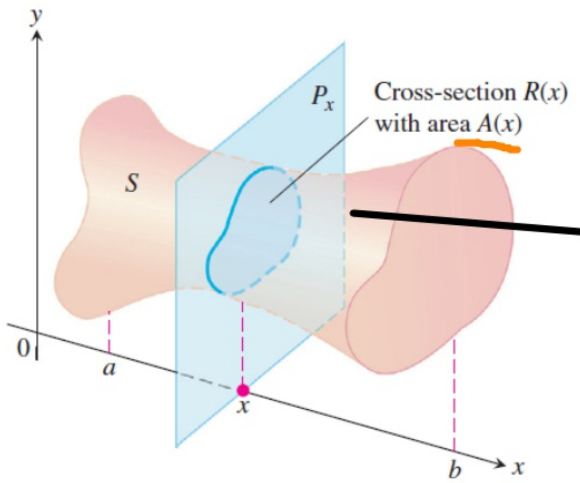


Figure 7.15 The cross section of an arbitrary solid at point x.

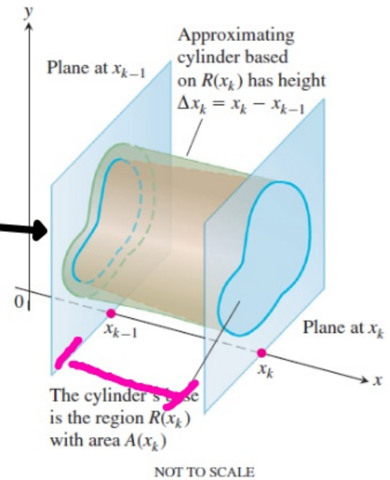


Figure 7.16 Enlarged view of the slice of the solid between the planes at x_{k-1} and x_k.

The volume of the cylinder is

$$V_k = \text{base area} \times \text{height} = A(x_k) \times \Delta x.$$

The sum

$$\sum V_k = \sum A(x_k) \times \Delta x$$

DEFINITION Volume of a Solid

The **volume of a solid** of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

How to Find Volume by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

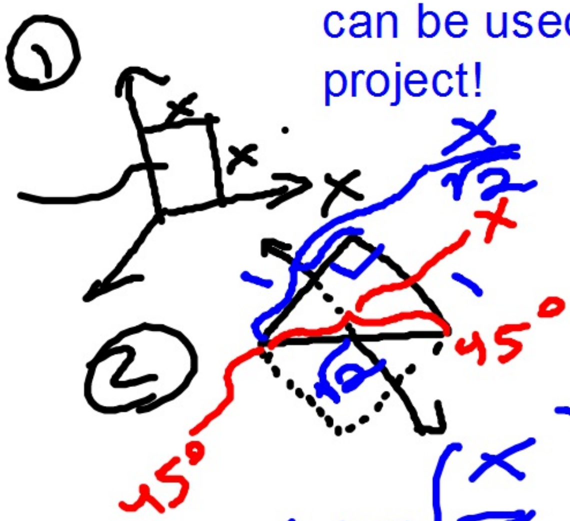
You'll want to write these down- these can be used for your project!

First, let's practice writing formula's for A(x)!

In Exercises 1–10, give a formula for the area of the plane region in terms of the single variable x.

1. a square with sides of length x x^2
2. a square with diagonals of length x $x^2/2$
3. a semicircle of radius x
4. a semicircle of diameter x
5. an equilateral triangle with sides of length x
6. an isosceles right triangle with legs of length x
7. an isosceles right triangle with hypotenuse x
8. an isosceles triangle with two sides of length $2x$ and one side of length x
9. a triangle with sides $3x$, $4x$, and $5x$
10. a regular hexagon with sides of length x

Area $= x \cdot x$



$$A(x) = \left(\frac{x}{\sqrt{2}}\right)^2$$

In Exercises 1–10, give a formula for the area of the plane region in terms of the single variable x.

1. a square with sides of length x $\quad x^2$

2. a square with diagonals of length x $\quad x^2/2$

3. a semicircle of radius x $\quad \pi x^2/2$

4. a semicircle of diameter x $\quad \pi x^2/8$

5. an equilateral triangle with sides of length x $\quad (\sqrt{3}/4)x^2$

3)

$A = \frac{1}{2}\pi r^2$

$A = \frac{1}{2}\pi x^2$

4

$r = \frac{1}{2}x$

$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(\frac{1}{2}$

$\frac{1}{8}\pi x^2$

In Exercises 1–10, give a formula for the area of the plane region in terms of the single variable x.

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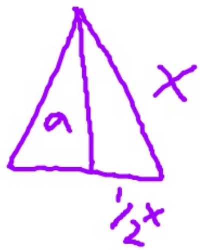
2. a square with diagonals of length x $x^2/2$
3. a semicircle of radius x $\pi x^2/2$
4. a semicircle of diameter x $\pi x^2/8$
5. an equilateral triangle with sides of length x $(\sqrt{3}/4)x^2$

5)



$$x^2 = \left(\frac{1}{2}x\right)^2 + a^2$$

$$x^2 = \frac{1}{4}x^2 + a^2$$

$$\frac{3}{4}x^2 = a^2$$

$$a = \sqrt{\frac{3}{4}x^2}$$

$$a = \frac{\sqrt{3}\,x}{2}$$

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(x)\left(\frac{\sqrt{3}\,x}{2}\right)$$

$$= \frac{\sqrt{3}\,x^2}{4}$$

6. an isosceles right triangle with legs of length x $x^2/2$

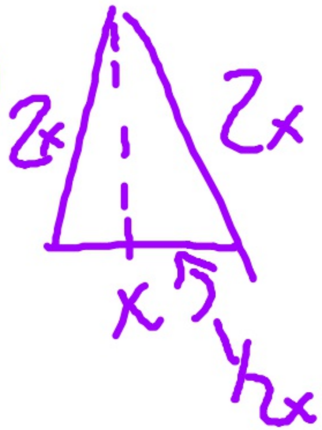
7. an isosceles right triangle with hypotenuse x $x^2/4$

8. an isosceles triangle with two sides of length $2x$ and one side of length x $(\sqrt{15}/4)x^2$

9. a triangle with sides $3x$, $4x$, and $5x$ $6x^2$

10. a regular hexagon with sides of length x $(3\sqrt{3}/2)x^2$

8



$A = \frac{1}{2}bh$

$A = \frac{1}{2}x\left(\frac{\sqrt{15}\,x}{2}\right)$

$A = \frac{\sqrt{15}}{4}x^2$

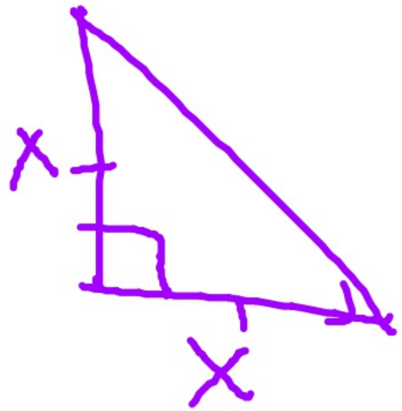
$(2x)^2 = \left(\frac{1}{2}x\right)^2 + c^2$

$4x^2 = \frac{1}{4}x^2 + c^2$

$\frac{15}{4}x^2 = c^2$

$c = \frac{\sqrt{15}\,x}{2}$

6



$A = \frac{1}{2}bh$

$A = \frac{1}{2}(x)(x)$

$A = \frac{1}{2}x^2$

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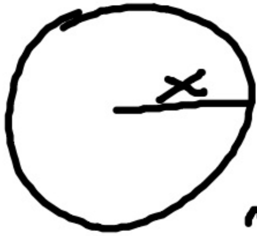
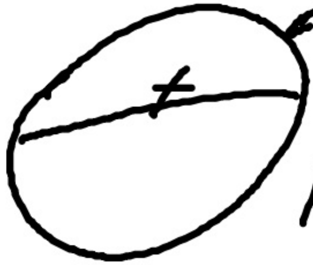
3



$A(x) = \frac{1}{2}\pi x^2$

4



$r = x/2$

$A(x) = \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

How to Find Volume by the Method of Slicing

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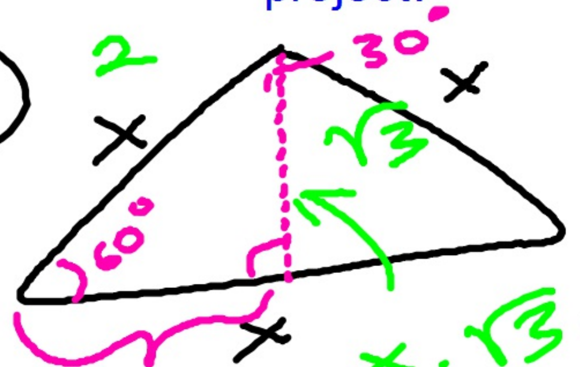
5



$x \cdot \sqrt{3} = h$

$b = x$

$A = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right)$

6. an isosceles right triangle with legs of length x $x^2/2$

7. an isosceles right triangle with hypotenuse x $x^2/4$

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9. a triangle with sides $3x$, $4x$, and $5x$ $6x^2$

10. a regular hexagon with sides of length x $(3\sqrt{3}/2)x^2$

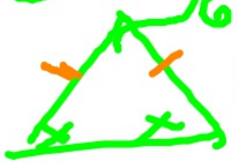
$P = 6x$

$A = \cancel{\frac{1}{2}}(\overset{3}{\cancel{6}}x)\left(\frac{\sqrt{3}}{2}x\right)$

$A = \frac{1}{2}Ph$

6-sided figures....

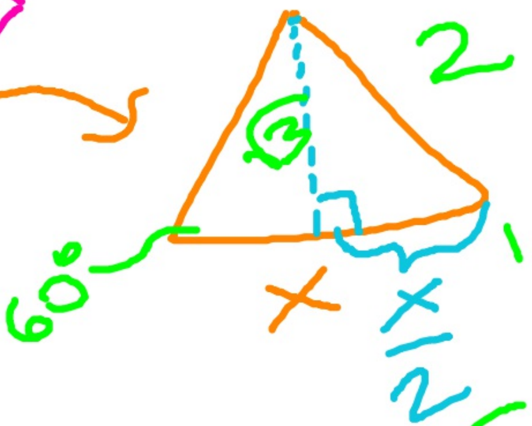
$\frac{360}{6} = 60$



$60 + 2x = 180$

$x = 60$

equilateral



$h = \frac{\sqrt{3}}{2}x$

Square Cross Sections

Let us apply the volume formula to a solid with square cross sections.

EXAMPLE 1 A Square-Based Pyramid

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

SOLUTION

We follow the steps for the method of slicing.

1. *Sketch.* We draw the pyramid with its vertex at the origin and its altitude along the interval $0 \le x \le 3$. We sketch a typical cross section at a point x between 0 and 3 (Figure 7.17).
2. *Find a formula for $A(x)$.* The cross section at x is a square x meters on a side, so
$$A(x) = x^2.$$
3. *Find the limits of integration.* The squares go from $x = 0$ to $x = 3$.
4. *Integrate to find the volume.*
$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2 = \frac{x^3}{3}\bigg]_0^3 = 9 \text{ m}^3$$

Now try Exercise 3.

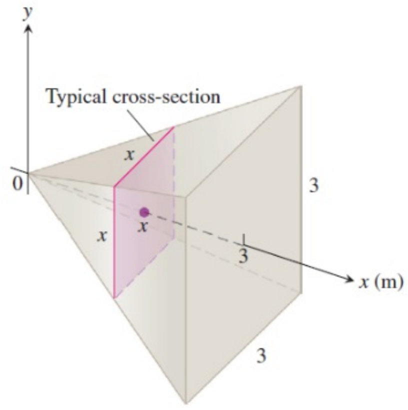


Figure 7.17 A cross section of the pyramid in Example 1.

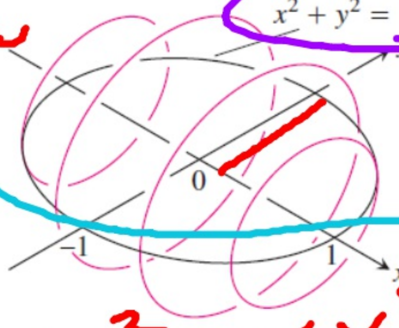
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross sections of the solid that are perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

(a) The cross sections are circular disks with diameters in the xy-plane. $\int_{-1}^{1} \pi(1 - x^2)\,dx$

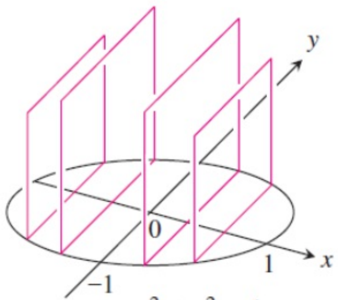
$x^2 + y^2 = 1 \rightarrow y^2 = 1 - x^2$

$y = \pm\sqrt{}$

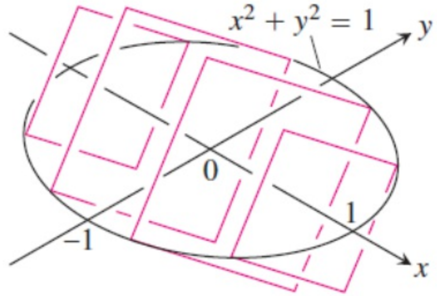


$$A = \pi r^2 = \pi y^2 = \pi\left(\sqrt{1 - x^2}\right)^2$$

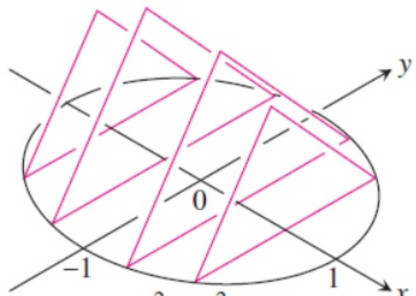
(b) The cross sections are squares with bases in the xy-plane. $\int_{-1}^{1} 4(1 - x^2)\,dx$



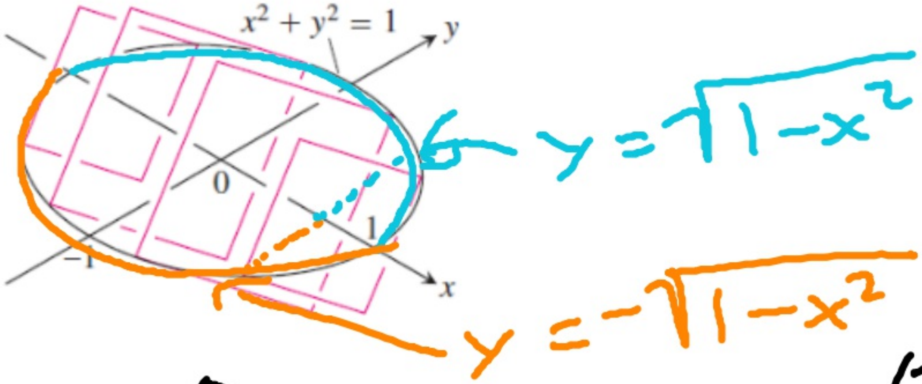
(c) The cross sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.) $\int_{-1}^{1} 2(1 - x^2)\,dx$

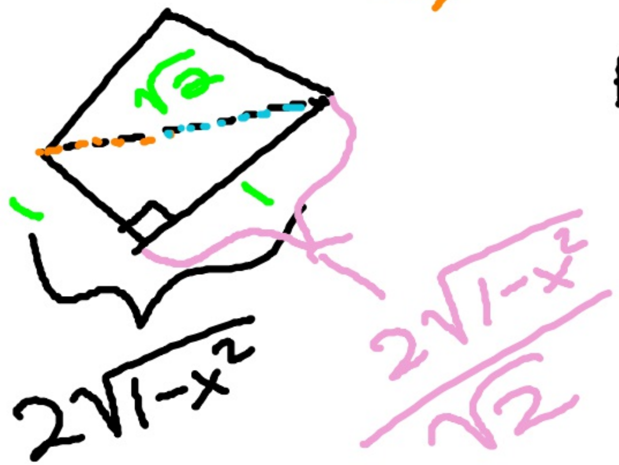


(d) The cross sections are equilateral triangles with bases in the xy-plane. $\int_{-1}^{1} \sqrt{3}(1 - x^2)\,dx$



(c) The cross sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.) $2(1 - x^2)$



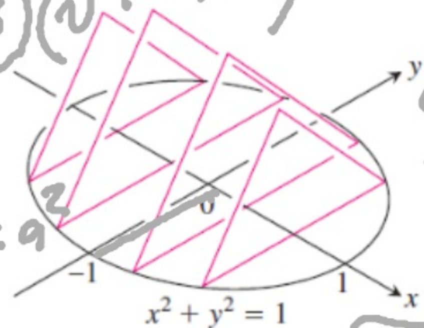


$$A(x) = \left(\frac{2\sqrt{1-x^2}}{\sqrt{2}}\right)^2$$

$$= \frac{4(1-x^2)}{2}$$

(d) The cross sections are equilateral triangles with bases in the xy-plane. $\sqrt{3}(1 - x^2)$

$a^2 = (\sqrt{3})(\sqrt{1-x^2})$



$(\sqrt{1-x^2})^2 + b^2 = a^2$

a

$2(\sqrt{1-x^2})$

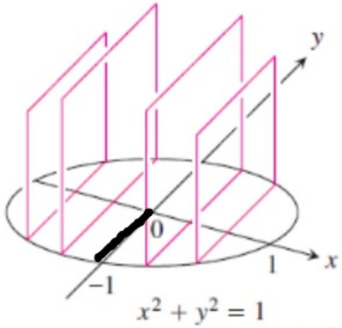
$y = \sqrt{1-x^2} \times \sqrt{3}$

2

$\sqrt{3}$

1

(b) The cross sections are squares with bases in the xy-plane. $4(1 - x^2)$



$y = \sqrt{1-x^2}$

$2(\sqrt{1-x^2})$

$A(x) = [2(\sqrt{1-x^2})]^2$

$= 4(1-x^2)$

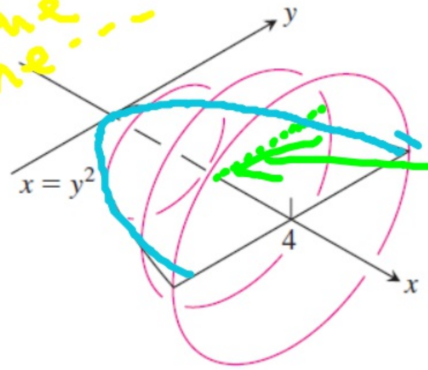
$A(x) = \cancel{2}(\sqrt{1-x^2}) \cdot (\sqrt{3})(\sqrt{1-x^2})$

$= \sqrt{3}(1-x^2)$

2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

Find the volume...

(a) The cross sections are circular disks with diameters in the xy-plane. $\int_0^4$ πx



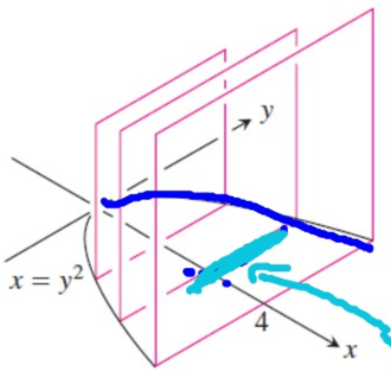
a) $r = y$

$r = \sqrt{x}$

$A = \pi r^2$

$A = \pi(\sqrt{x})^2$

(b) The cross sections are squares with bases in the xy-plane. $\int_0^4$ $4x$



$\sqrt{x} + \sqrt{x} = (2\sqrt{x})^2$

(c) The cross sections are squares with diagonals in the xy-plane. $\int_0^4$ $2x$

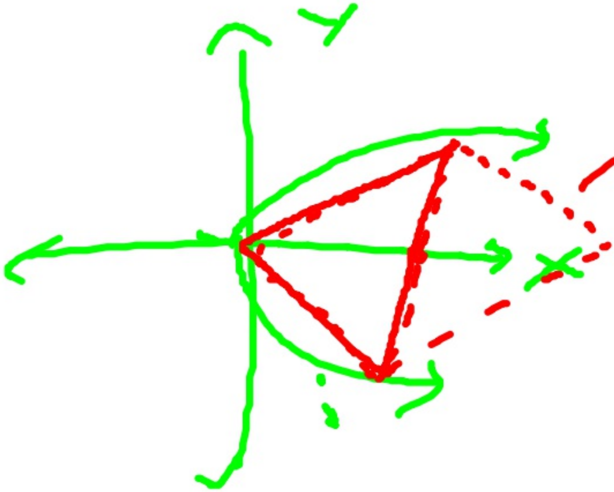
(d) The cross sections are equilateral triangles with bases in the xy-plane. $\int_0^4$ $\sqrt{3}x$

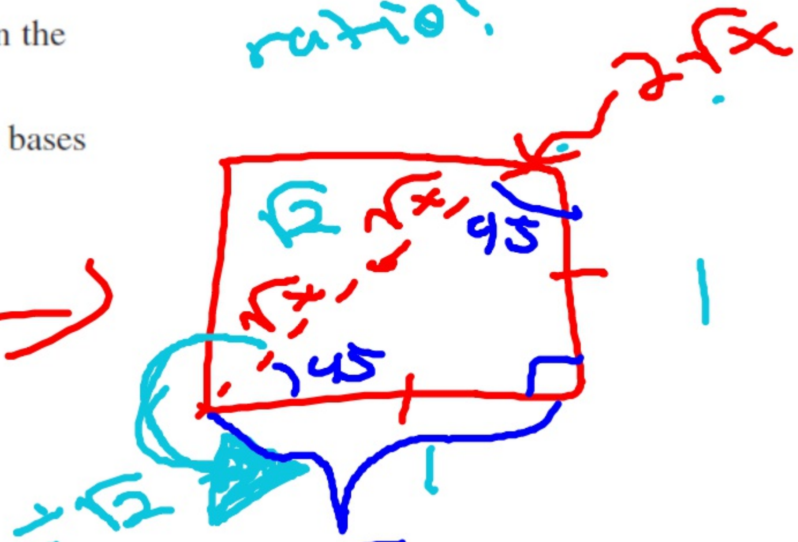
2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

(c) The cross sections are squares with diagonals in the xy-plane. $2x$

(d) The cross sections are equilateral triangles with bases in the xy-plane. $\sqrt{3}x$

45°-45°-90° ratio!





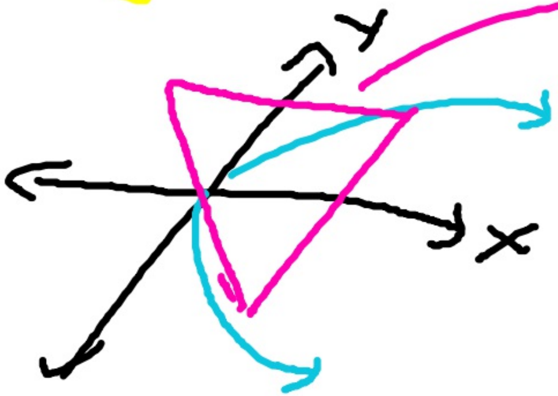
$$s = \frac{2\sqrt{x}}{\sqrt{2}}$$

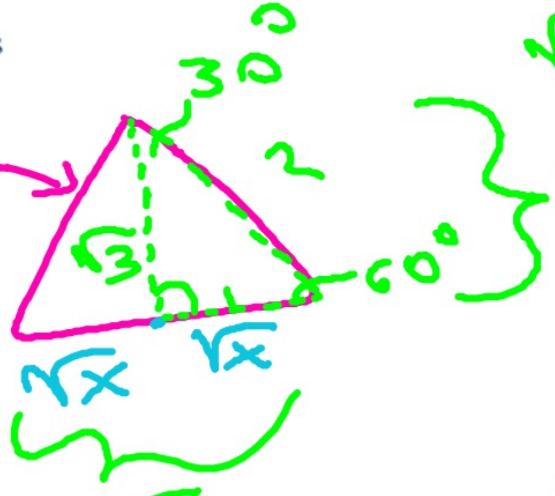
$$s^2 = \frac{4x}{2} = 2x$$

2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

(c) The cross sections are squares with diagonals in the xy-plane. $\int_0^4$ $2x$

(d) The cross sections are equilateral triangles with bases in the xy-plane $\int_0^4$ $\sqrt{3}x$





$\sqrt{3} \cdot \sqrt{x}$

$\sqrt{3x}$

$2\sqrt{x}$

$A = \cancel{\frac{1}{2}}(\cancel{2}\sqrt{x})(\sqrt{3x})$

$= \sqrt{3x^2}$

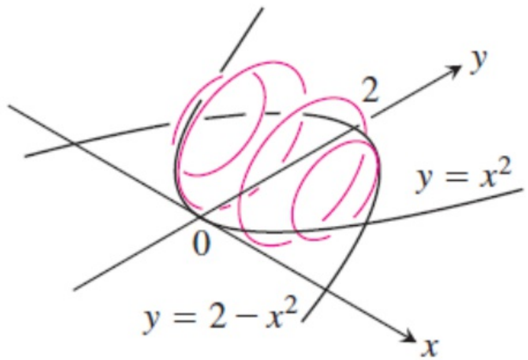
$A = x\sqrt{3}$

$\sqrt{3} \cdot x$

In Exercises 3–6, find the volume of the solid analytically.

3. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from $y = -\sqrt{x}$ to $y = \sqrt{x}$. 16

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. $16\pi/15$

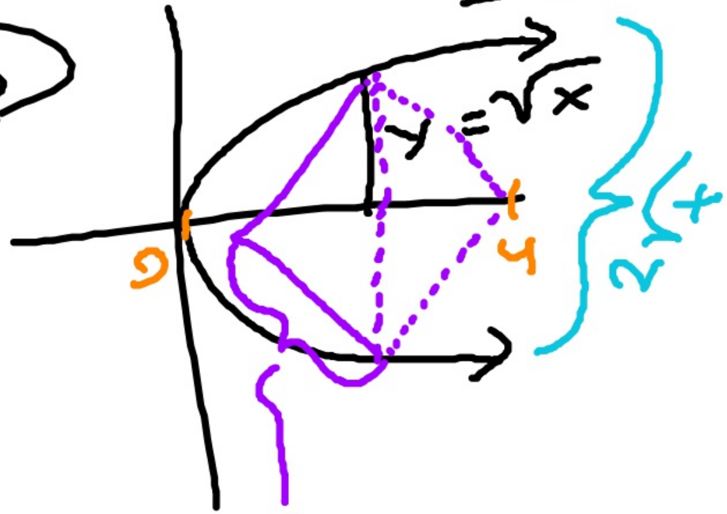


$x = y^2$

$A(x) = \left(\frac{2\sqrt{x}}{\sqrt{2}}\right)^2$

$= 2x$

(3)

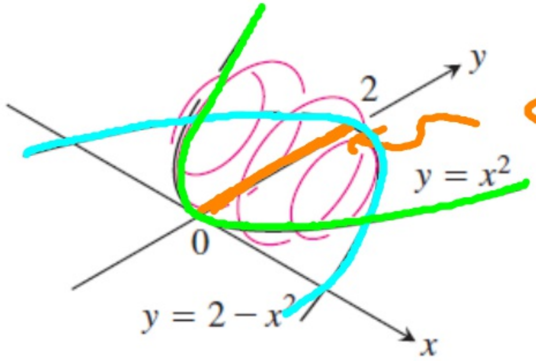


$\frac{2\sqrt{x}}{\sqrt{2}}$

$\int_0^4 2x\,dx$

$= x^2\Big|_0^4 = (4)^2 - (0)^2 = 16$

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. $16\pi/15$



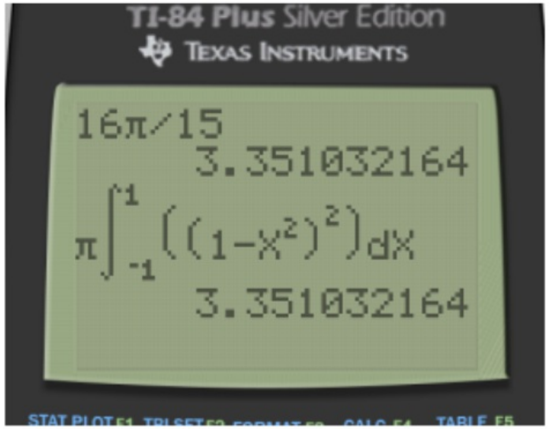
$$\text{diameter} = (2 - x^2) - x^2$$
$$= 2 - 2x^2$$
$$\text{radius} = \frac{2 - 2x^2}{2}$$

$$A(x) = \pi r^2$$
$$r = 1 - x^2$$

so...

$$A(x) = \pi(1 - x^2)^2$$

$$V = \pi \int_{-1}^{1} (1 - x^2)^2$$



5. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

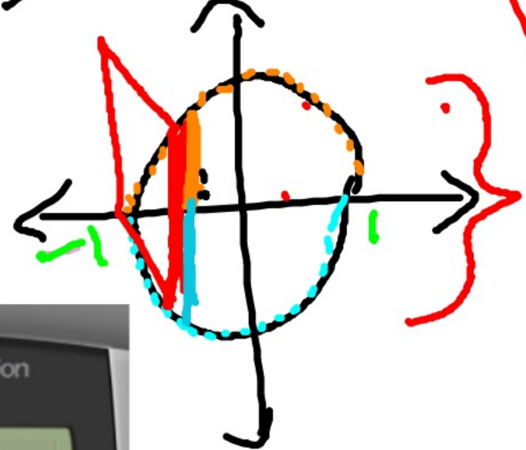
16/3

$$\int_a^b A(x)\,dx$$

$$(y)^2 = \left(\sqrt{1-x^2}\right)^2$$

$$y^2 = 1 - x^2$$

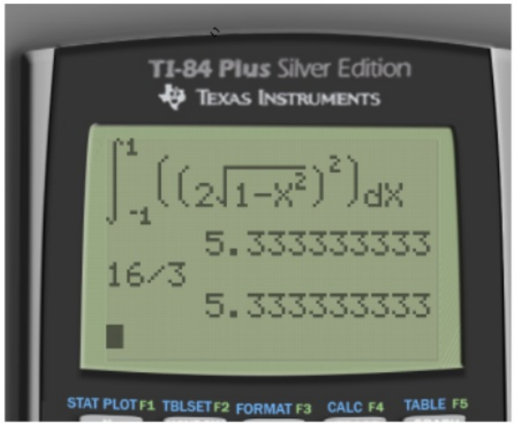
$$x^2 + y^2 = 1$$



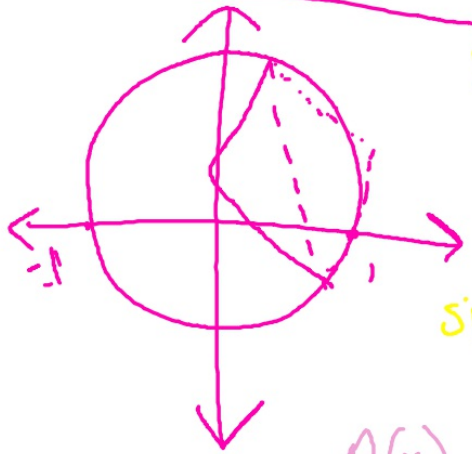
base length $= 2\sqrt{1-x^2}$

$$A(x) = \left(2\sqrt{1-x^2}\right)^2$$

$$V = \int_{-1}^{1} \left(2\sqrt{1-x^2}\right)^2 dx$$



6. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. 8/3



$2\sqrt{1-x^2}$

side $= \dfrac{2\sqrt{1-x^2}}{\sqrt{2}}$

$$A(x) = \left(\frac{2\sqrt{1-x^2}}{\sqrt{2}}\right)^2$$

$$= 2(1-x^2)$$

$$V = \int_{-1}^{1} 2(1-x^2)\,dx$$

$$= 2.66 = 8/3$$