What you'll learn about

- Volume As an Integral
- Square Cross Sections
- Circular Cross Sections
- Cylindrical Shells
- Other Cross Sections

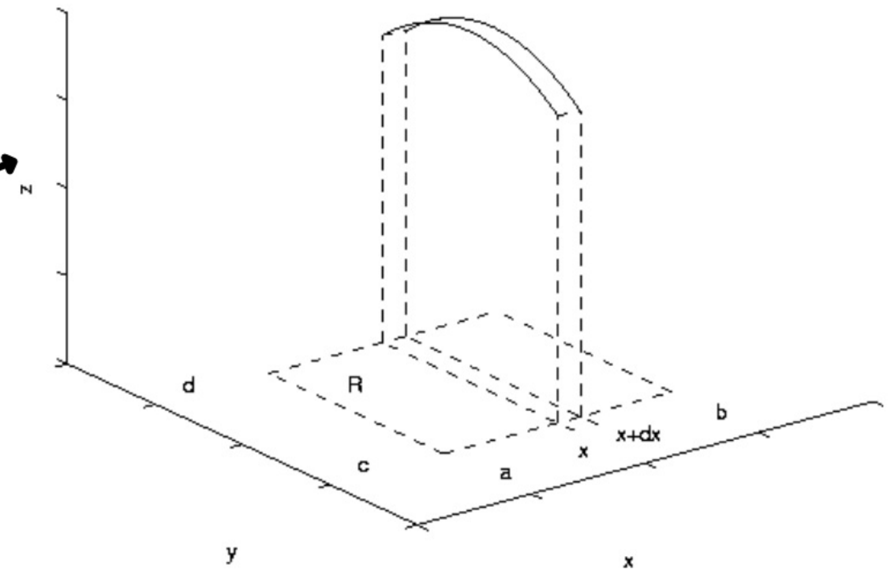
. . . and why

The techniques of this section allow us to compute volumes of certain solids in three dimensions.

7.3 Volumes

In this section, we are going to generate volumes with known cross sections.

First, let's think of it like a loaf of sliced bread. We will find the volume of each "slice," and then add them all up as the thickness approaches 0 (integrate!)



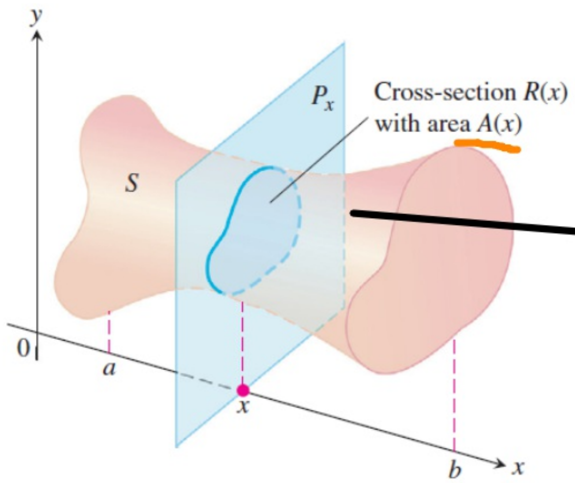


Figure 7.15 The cross section of an arbitrary solid at point x.

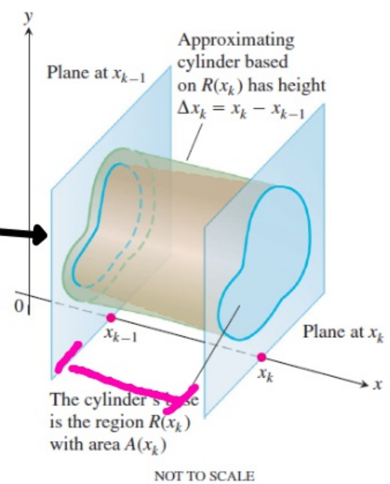


Figure 7.16 Enlarged view of the slice of the solid between the planes at x_{k-1} and x_k.

The volume of the cylinder is

$$V_k = \text{base area} \times \text{height} = A(x_k) \times \Delta x.$$

The sum

$$\sum V_k = \sum A(x_k) \times \Delta x$$

DEFINITION Volume of a Solid

The **volume of a solid** of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

How to Find Volume by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

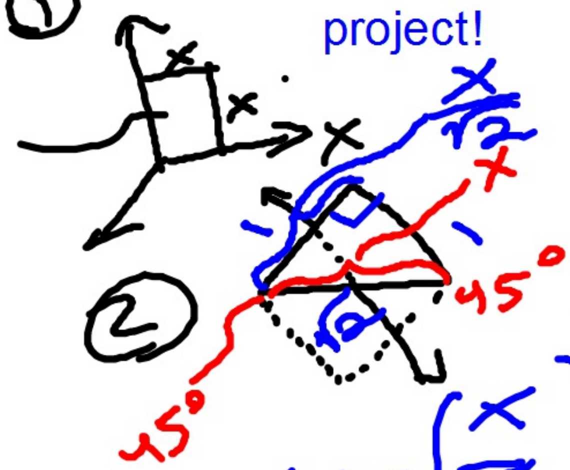
You'll want to write these down- these can be used for your project!

First, let's practice writing formula's for A(x)!

In Exercises 1–10, give a formula for the area of the plane region in terms of the single variable x.

1. a square with sides of length x x^2
2. a square with diagonals of length x $x^2/2$
3. a semicircle of radius x
4. a semicircle of diameter x
5. an equilateral triangle with sides of length x
6. an isosceles right triangle with legs of length x
7. an isosceles right triangle with hypotenuse x
8. an isosceles triangle with two sides of length $2x$ and one side of length x
9. a triangle with sides $3x$, $4x$, and $5x$
10. a regular hexagon with sides of length x

area $= x \cdot x$



$A(x) = \left(\frac{x}{\sqrt{2}}\right)^2$

In Exercises 1–10, give a formula for the area of the plane region in terms of the single variable x.

1. a square with sides of length x x^2
2. a square with diagonals of length x $x^2/2$
3. a semicircle of radius x $\pi x^2/2$
4. a semicircle of diameter x $\pi x^2/8$
5. an equilateral triangle with sides of length x $(\sqrt{3}/4)x^2$

3)

$A = \frac{1}{2}\pi r^2$

$A = \frac{1}{2}\pi x^2$

4

$r = \frac{1}{2}x$

$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(\frac{1}{2}$

$\frac{1}{8}\pi x^2$

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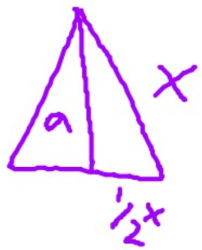


$$x^2 = \left(\tfrac{1}{2}x\right)^2 + a^2$$

$$x^2 = \frac{1}{4}x^2 + a^2$$

$$\tfrac{3}{4}x^2 = a^2$$

$$a = \sqrt{\tfrac{3}{4}x^2}$$

$$a = \frac{\sqrt{3}\,x}{2}$$

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(x)\left(\frac{\sqrt{3}\,x}{2}\right)$$

$$= \frac{\sqrt{3}\,x^2}{4}$$

6. an isosceles right triangle with legs of length x $x^2/2$

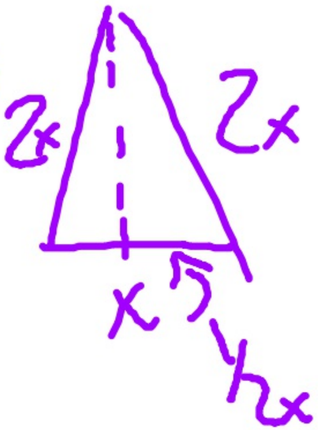
7. an isosceles right triangle with hypotenuse x $x^2/4$

8. an isosceles triangle with two sides of length $2x$ and one side of length x $(\sqrt{15}/4)x^2$

9. a triangle with sides $3x$, $4x$, and $5x$ $6x^2$

10. a regular hexagon with sides of length x $(3\sqrt{3}/2)x^2$

8



$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}x\left(\sqrt{15}\,x\right)$$

$$A = \frac{\sqrt{15}}{4}x^2$$

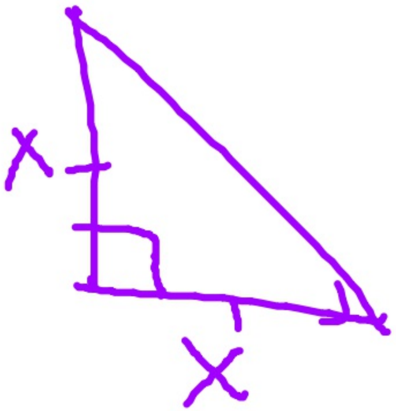
$$(2x)^2 = \left(\frac{1}{2}x\right)^2 + c^2$$

$$4x^2 = \frac{1}{4}x^2 + c^2$$

$$\frac{15}{4}x^2 = c^2$$

$$c = \frac{\sqrt{15}\,x}{2}$$

6



$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(x)(x)$$

$$A = \frac{1}{2}x^2$$

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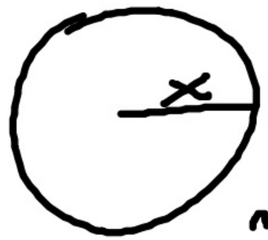
3



$A(x) = \frac{1}{2}\pi x^2$

4

$r = \frac{x}{2}$

$A(x) = \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

How to Find Volume by the Method of Slicing

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2. Find a formula for $A(x)$.
3. Find the limits of integration.
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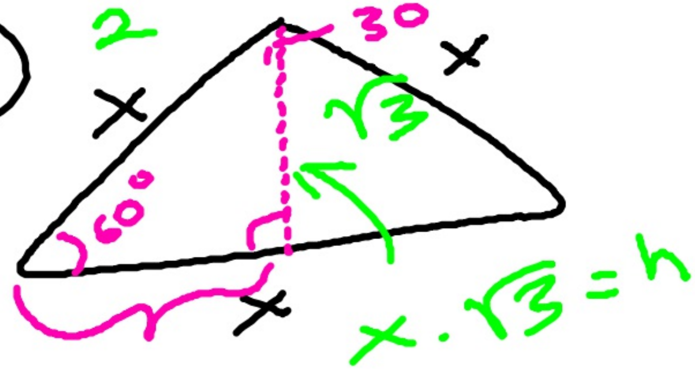
5



$b = x$

$A = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right)$

6. an isosceles right triangle with legs of length x $\quad x^2/2$

7. an isosceles right triangle with hypotenuse x $\quad x^2/4$

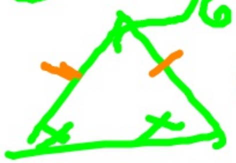
8. an isosceles triangle with two sides of length $2x$ and one side of length x $\quad (\sqrt{15}/4)x^2$

9. a triangle with sides $3x$, $4x$, and $5x$ $\quad 6x^2$

10. a regular hexagon with sides of length x $\quad (3\sqrt{3}/2)x^2$

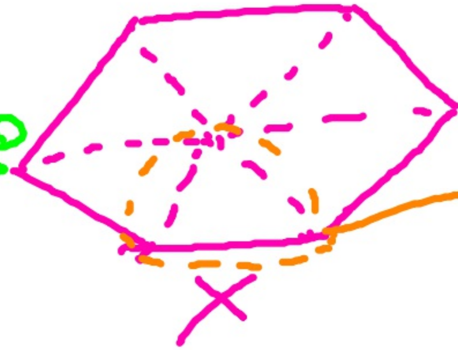
6-sided figures....

$\frac{360}{6} = 60$



$60 + 2x = 180$

equilateral $x = 60$



$P = 6x$

$A = \frac{1}{\cancel{2}} (\cancel{6}^{3} x)\left(\frac{\sqrt{3}}{2} x\right)$

$A = \frac{1}{2} P \underline{h}$

$h = \frac{\sqrt{3}}{2} x$

Square Cross Sections

Let us apply the volume formula to a solid with square cross sections.

EXAMPLE 1 A Square-Based Pyramid

A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

SOLUTION

We follow the steps for the method of slicing.

1. *Sketch.* We draw the pyramid with its vertex at the origin and its altitude along the interval $0 \le x \le 3$. We sketch a typical cross section at a point x between 0 and 3 (Figure 7.17).
2. *Find a formula for $A(x)$.* The cross section at x is a square x meters on a side, so
$$A(x) = x^2.$$
3. *Find the limits of integration.* The squares go from $x = 0$ to $x = 3$.
4. *Integrate to find the volume.*
$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2 = \frac{x^3}{3}\Big]_0^3 = 9 \text{ m}^3$$

Now try Exercise 3.

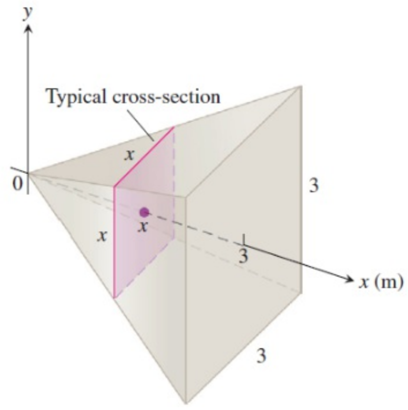
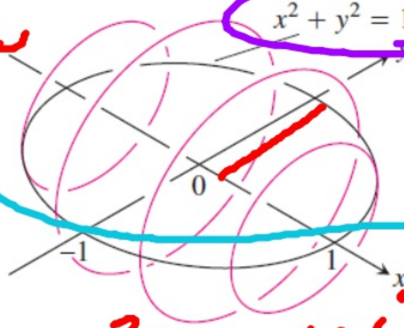


Figure 7.17 A cross section of the pyramid in Example 1.

In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross sections of the solid that are perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

(a) The cross sections are circular disks with diameters in the xy-plane. $\int_{-1}^{1} \pi(1 - x^2)\, dx$

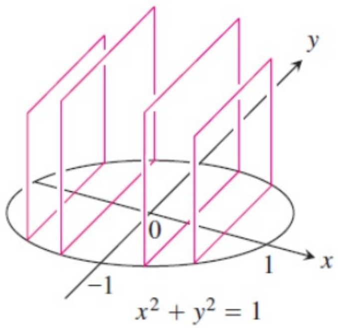


$y^2 = 1 - x^2$

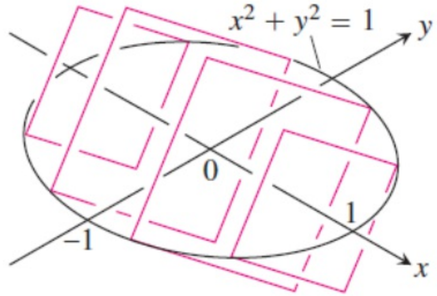
$y = \pm\sqrt{}$

$A = \pi r^2 = \pi y^2$
$= \pi(\sqrt{1-x^2})^2$

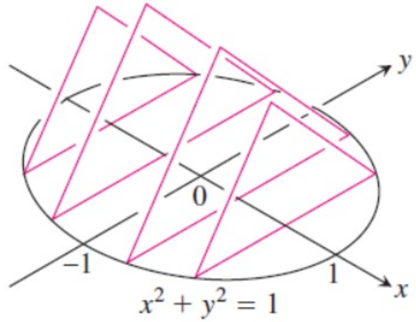
(b) The cross sections are squares with bases in the xy-plane. $\int_{-1}^{1} 4(1 - x^2)\, dx$



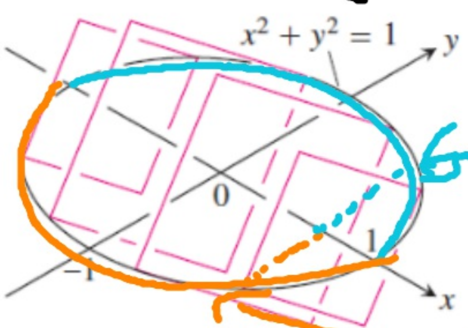
(c) The cross sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.) $\int_{-1}^{1} 2(1 - x^2)\, dx$



(d) The cross sections are equilateral triangles with bases in the xy-plane. $\int_{-1}^{1} \sqrt{3}(1 - x^2)\, dx$

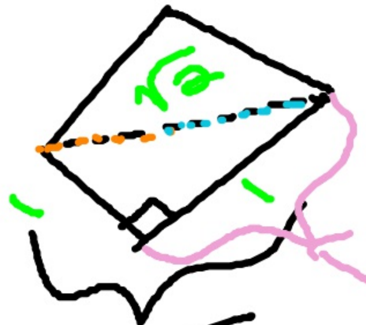


(c) The cross sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.) $2(1 - x^2)$



$y = \sqrt{1-x^2}$

$y = -\sqrt{1-x^2}$



$2\sqrt{1-x^2}$

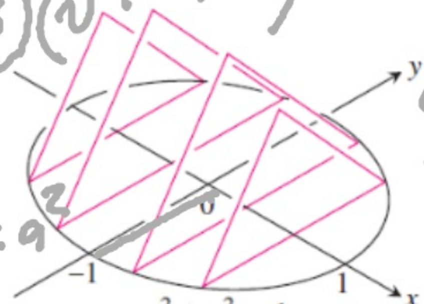
$\frac{2\sqrt{1-x^2}}{\sqrt{2}}$

$$A(x) = \left(\frac{2\sqrt{1-x^2}}{\sqrt{2}}\right)^2$$

$$= \frac{4(1-x^2)}{2}$$

(d) The cross sections are equilateral triangles with bases in the xy-plane. $\sqrt{3}(1-x^2)$

$a^2 = (\sqrt{3})(\sqrt{1-x^2})$



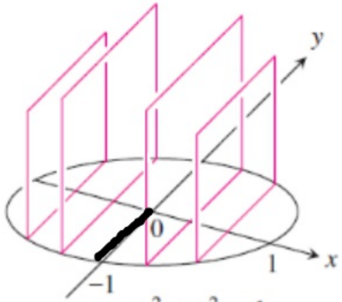
$x^2 + y^2 = 1$

$(\sqrt{1-x^2})^2 + b^2 = a^2$

$2(\sqrt{1-x^2})$

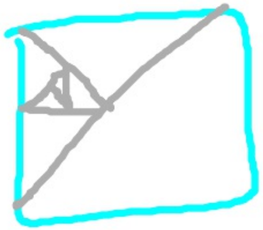
$y = \sqrt{1-x^2} \times \sqrt{3}$

(b) The cross sections are squares with bases in the xy-plane. $4(1-x^2)$



$x^2 + y^2 = 1$

$y = \sqrt{1-x^2}$



$2(\sqrt{1-x^2})$

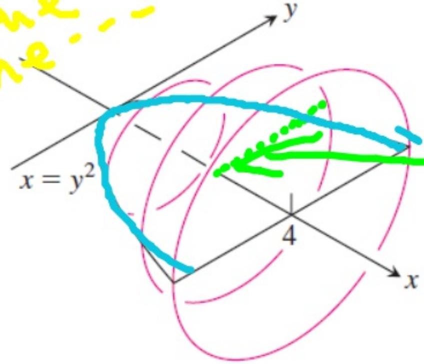
$$A(x) = [2(\sqrt{1-x^2})]^2 = 4(1-x^2)$$

$$A(x) = \cancel{2}(\sqrt{1-x^2}) \cdot (\sqrt{3})(\sqrt{1-x^2}) = \sqrt{3}(1-x^2)$$

2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

Find the volume...

(a) The cross sections are circular disks with diameters in the xy-plane. $\int_0^4$ πx



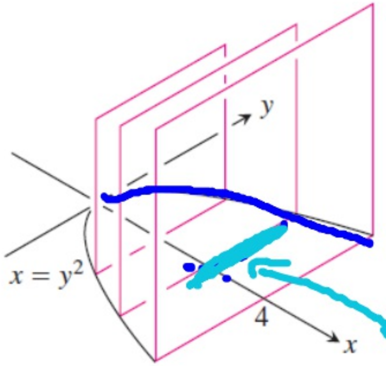
a) $r = y$

$r = \sqrt{x}$

$A = \pi r^2$

$A = \pi(\sqrt{x})^2$

(b) The cross sections are squares with bases in the xy-plane. $\int_0^4$ $4x$



$\sqrt{x} + \sqrt{x} = (2\sqrt{x})^2$

(c) The cross sections are squares with diagonals in the xy-plane. $\int_0^4$ $2x$

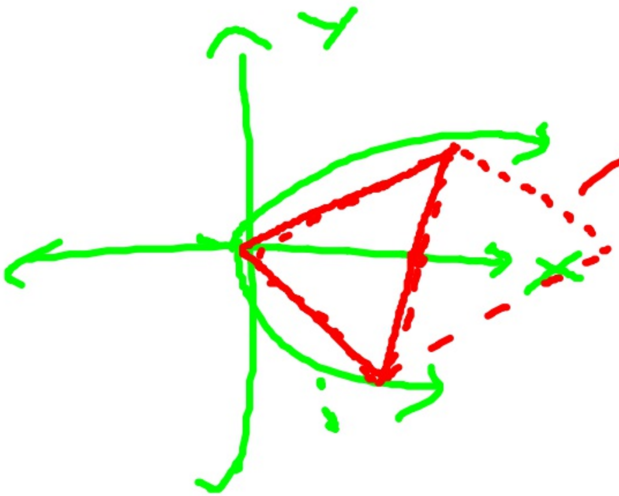
(d) The cross sections are equilateral triangles with bases in the xy-plane. $\int_0^4$ $\sqrt{3}x$

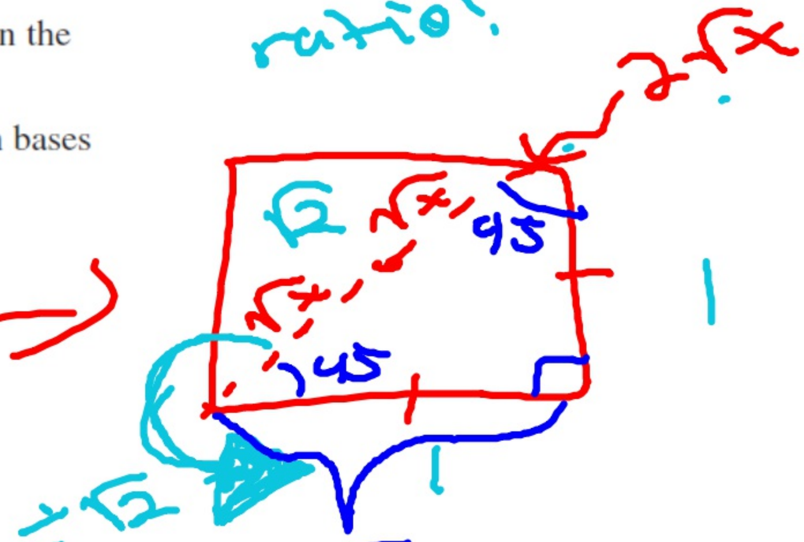
2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

45°-45°-90° ratio!

(c) The cross sections are squares with diagonals in the xy-plane. $2x$

(d) The cross sections are equilateral triangles with bases in the xy-plane. $\sqrt{3}x$





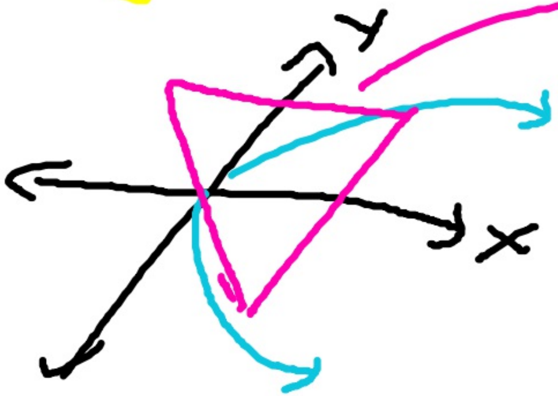
$$s = \frac{2\sqrt{x}}{\sqrt{2}}$$

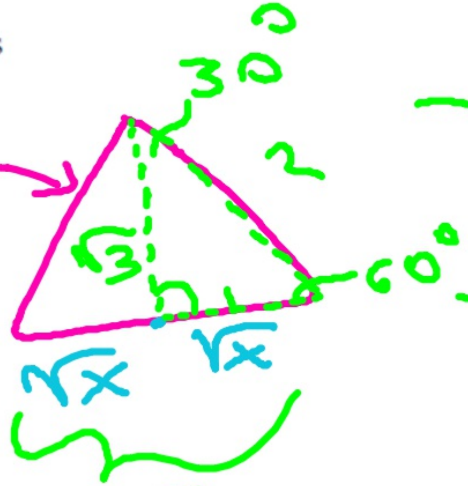
$$s^2 = \frac{4x}{2} = 2x$$

2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

(c) The cross sections are squares with diagonals in the xy-plane. $\int_0^4$ $2x$

(d) The cross sections are equilateral triangles with bases in the xy-plane $\int_0^4$ $\sqrt{3}x$





$\sqrt{3}\cdot\sqrt{x}$

$\sqrt{3x}$

$2\sqrt{x}$

$A = \cancel{\frac{1}{2}}(\cancel{2}\sqrt{x})(\sqrt{3x})$

$= \sqrt{3x^2}$

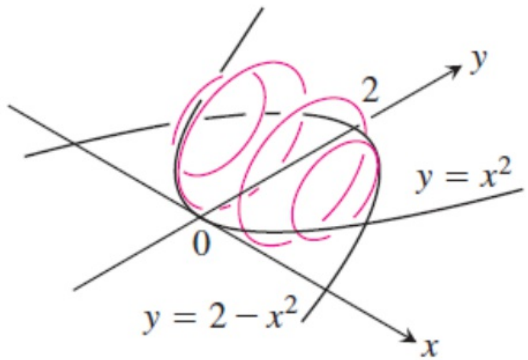
$A = x\sqrt{3}$

$\sqrt{3}\cdot x$

In Exercises 3–6, find the volume of the solid analytically.

3. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from $y = -\sqrt{x}$ to $y = \sqrt{x}$. 16

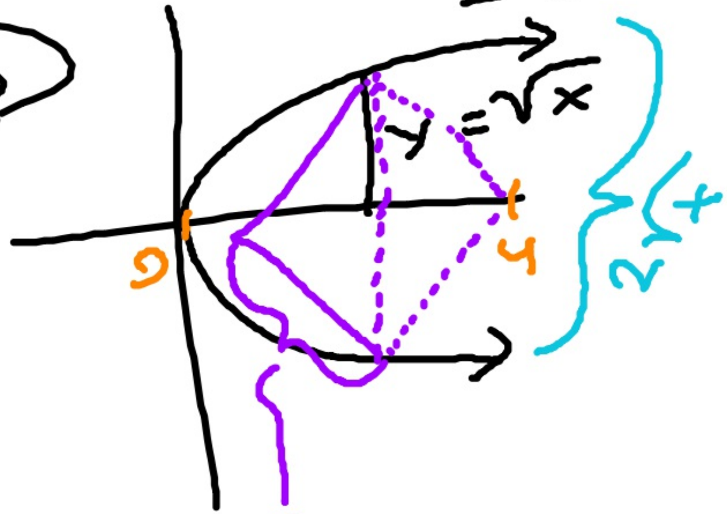
4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. $16\pi/15$



$x = y^2$

$A(x) = \left(\frac{2\sqrt{x}}{\sqrt{2}}\right)^2$
$= 2x$

3

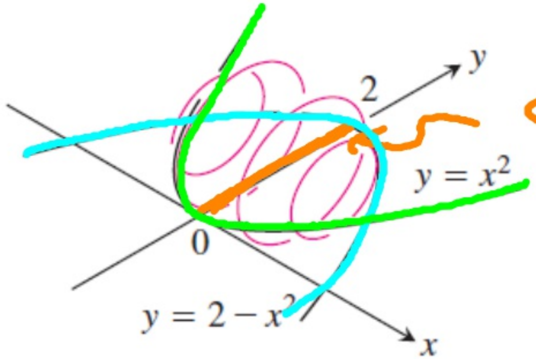


$\frac{2\sqrt{x}}{\sqrt{2}}$

$\int_0^4 2x\,dx$

$= x^2\Big|_0^4 = (4)^2 - (0)^2 = 16$

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. $16\pi/15$



$$\text{diameter} = (2 - x^2) - x^2 = 2 - 2x^2$$

$$\text{radius} = \frac{2 - 2x^2}{2}$$

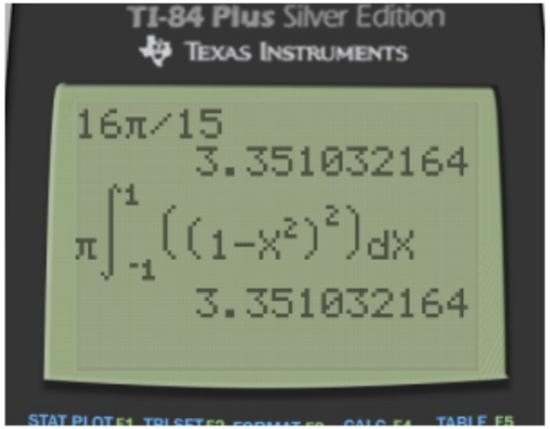
$$A(x) = \pi r^2$$

$$r = 1 - x^2$$

so...

$$A(x) = \pi(1 - x^2)^2$$

$$V = \pi \int_{-1}^{1} (1 - x^2)^2$$



5. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

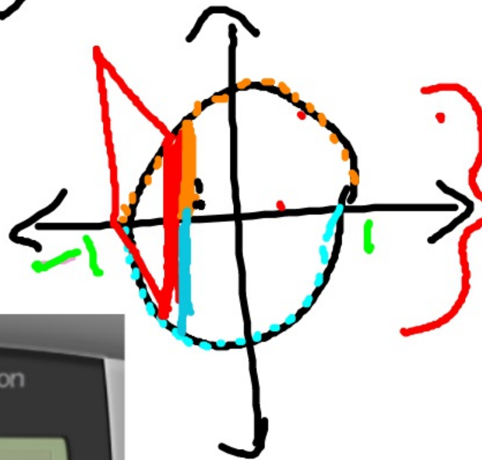
16/3

$$\int_a^b A(x)\,dx$$

$$(y)^2 = \left(\sqrt{1-x^2}\right)^2$$

$$y^2 = 1 - x^2$$

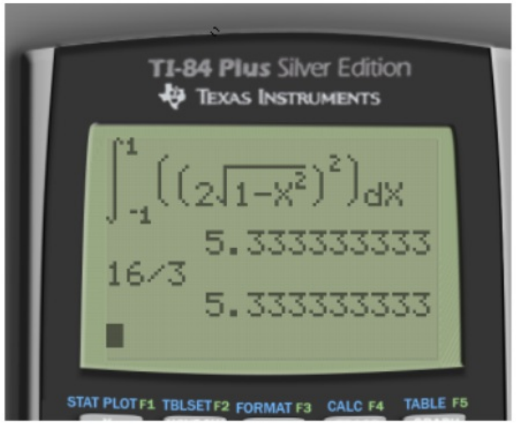
$$x^2 + y^2 = 1$$



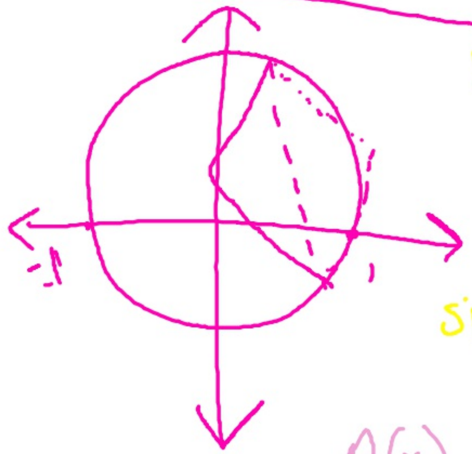
base length $= 2\sqrt{1-x^2}$

$$A(x) = \left(2\sqrt{1-x^2}\right)^2$$

$$V = \int_{-1}^{1} \left(2\sqrt{1-x^2}\right)^2 dx$$



6. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. 8/3



$\to 2\sqrt{1-x^2}$

side $= \dfrac{2\sqrt{1-x^2}}{\sqrt{2}}$

$$A(x) = \left(\frac{2\sqrt{1-x^2}}{\sqrt{2}}\right)^2$$

$$= 2(1-x^2)$$

$$V = \int_{-1}^{1} 2(1-x^2)\,dx$$

$$= 2.66 = 8/3$$

Circular Cross Sections

The only thing that changes when the cross sections of a solid are circular is the formula for $A(x)$. Many such solids are **solids of revolution,** as in the next example.

EXAMPLE 2 A Solid of Revolution

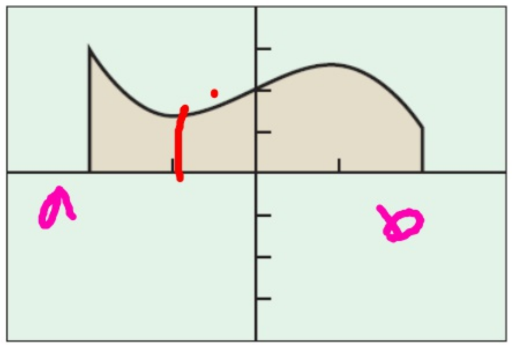
The region between the graph of $f(x) = 2 + x\cos x$ and the x-axis over the interval $[-2, 2]$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

SOLUTION

Revolving the region (Figure 7.18) about the x-axis generates the vase-shaped solid in Figure 7.19. The cross section at a typical point x is circular, with radius equal to $f(x)$. Its area is

$$A(x) = \pi(f(x))^2.$$

continued



[–3, 3] by [–4, 4]

Figure 7.18 The region in Example 2.

rotate around the x-axis.

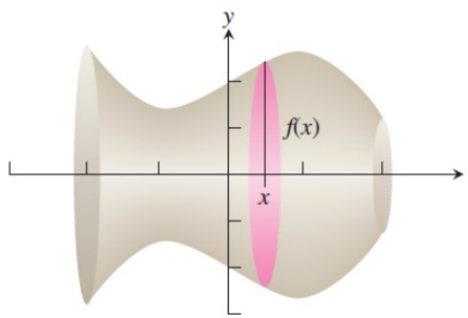


Figure 7.19 The region in Figure 7.18 is revolved about the x-axis to generate a solid. A typical cross section is circular, with radius $f(x) = 2 + x\cos x$. (Example 2)

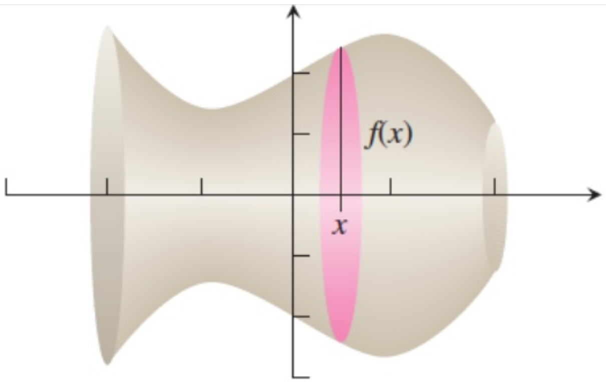


Figure 7.19 The region in Figure 7.18 is revolved about the x-axis to generate a solid. A typical cross section is circular, with radius $f(x) = 2 + x \cos x$. (Example 2)

We're adding up each "disc," where each disc is an area of a circle...

The volume of the solid is

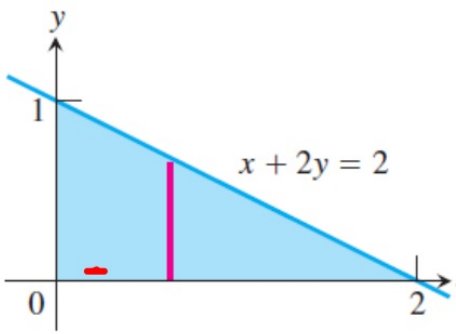
$$V = \int_{-2}^{2} A(x)\,dx$$

$$\approx \text{NINT}\,(\pi(2 + x\cos x)^2, x, -2, 2) \approx 52.43 \text{ units cubed.}$$

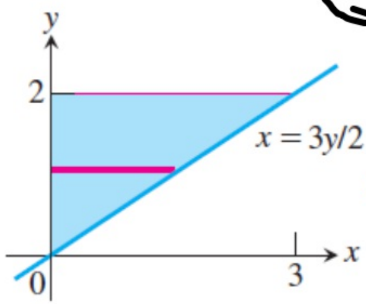
Now try Exercise 7.

In Exercises 7–10, find the volume of the solid generated by revolving the shaded region about the given axis.

7. about the x-axis $2\pi/3$



8. about the y-axis 6π



7

$x + 2y = 2$

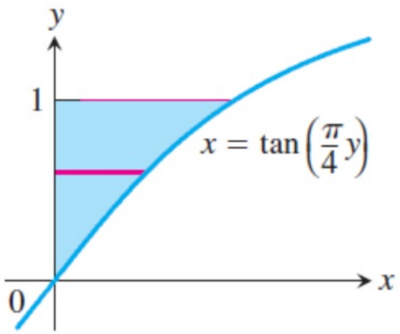
$y = \left(\frac{2-x}{2}\right)$

$\pi \int_0^2 \left(\frac{2-x}{2}\right)^2 dx$

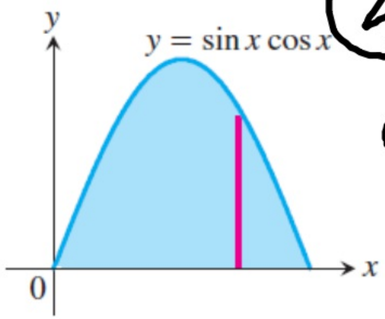
8

$\pi \int_0^2 \left(\frac{3y}{2}\right)^2 dy$

9. about the y-axis $4 - \pi$



10. about the x-axis $\pi^2/16$



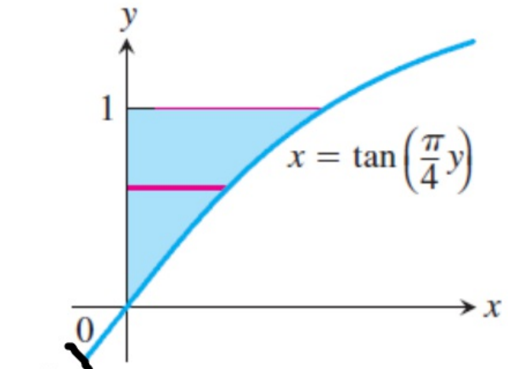
7 →

```
π∫₀²(((2-X)/2)²)d▸
                2.094395102
2π/3
                2.094395102
```

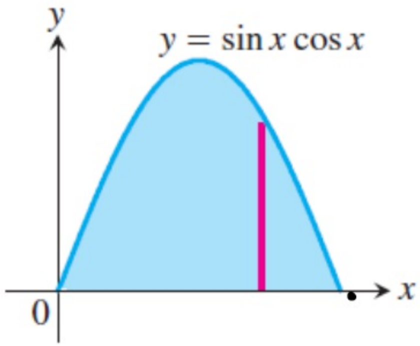
```
6π
                18.84955592
π∫₀²((3X/2)²)dX
                18.84955592
```

Look at #8 and 9. We can revolve the region around the x-axis, but notice that a "gap" in the solid will be generated...

9. about the y-axis $4 - \pi$



10. about the x-axis $\pi^2/16$



9)

$$\pi \int_0^1 \tan\left(\frac{\pi}{4} y\right)^2 dy$$

$= 4 - \pi$

$= .858$

10

$$\pi \int_0^{\pi/2} (\sin x \cos x)^2 d$$

$= .6168$

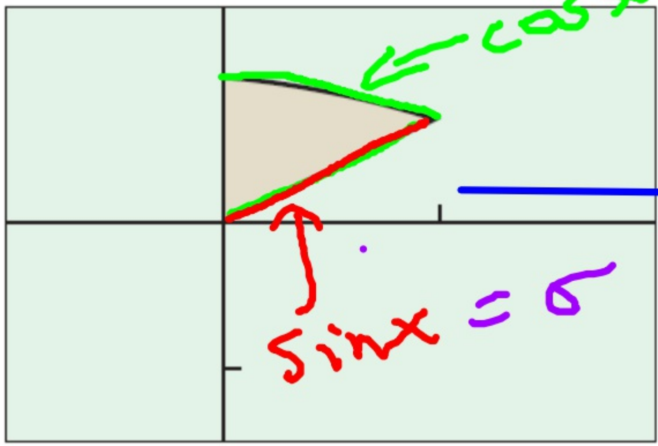
$= \pi^2/16$

EXAMPLE 3 Washer Cross Sections

The region in the first quadrant enclosed by the y-axis and the graphs of $y = \cos x$ and $y = \sin x$ is revolved about the x-axis to form a solid. Find its volume.

SOLUTION

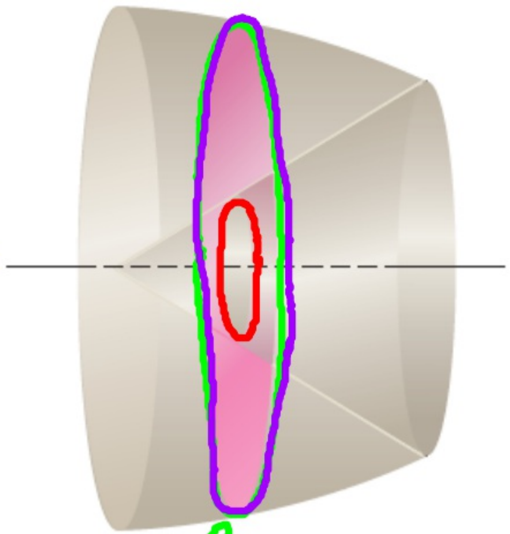
The region is shown in Figure 7.20.



$[-\pi/4, \pi/2]$ by $[-1.5, 1.5]$

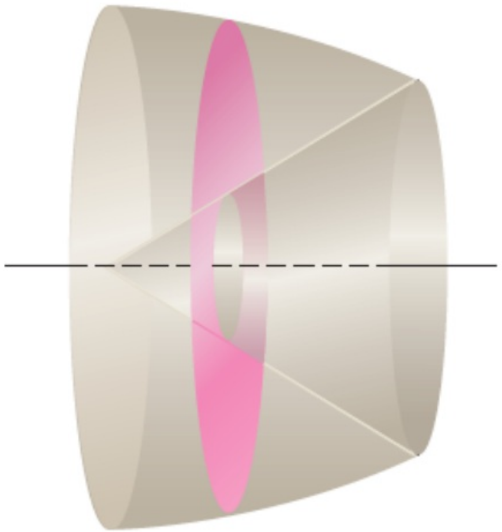
Figure 7.20 The region in Example 3.

revolve around the x-axis...



$\pi R^2 - \pi r^2$

Figure 7.21 The solid generated by revolving the region in Figure 7.20 about the x-axis. A typical cross section is a washer: a circular region with a circular region cut out of its center. (Example 3)



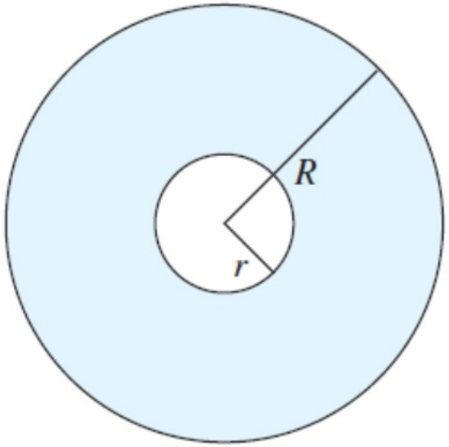
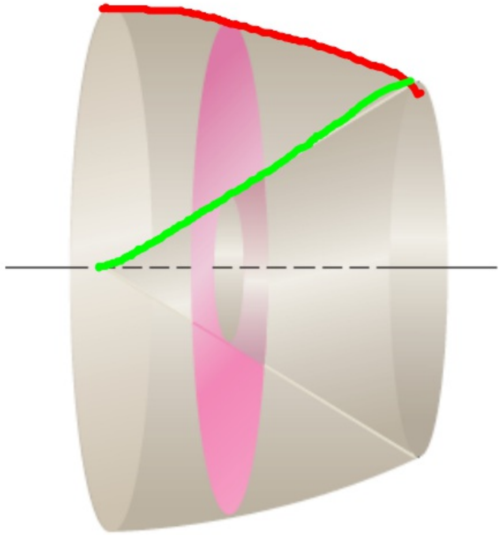
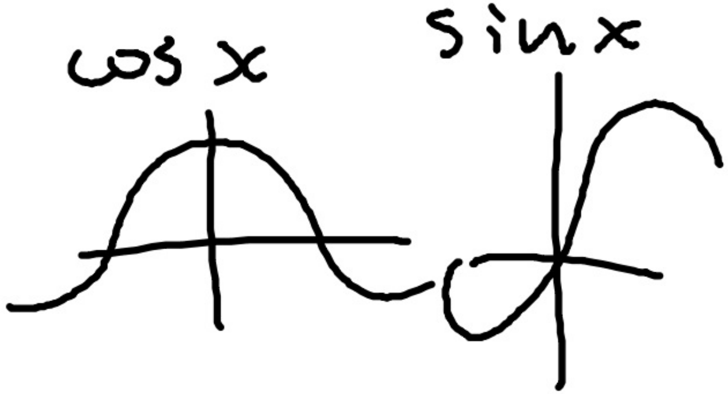


Figure 7.22 The area of a washer is $\pi R^2 - \pi r^2$. (Example 3)

We will use this to write our integration...





In our region the cosine curve defines the outer radius, and the curves intersect at $x = \pi/4$. The volume is

$$V = \int_0^{\pi/4} \pi (\cos^2 x - \sin^2 x)\, dx$$

$$= \pi \int_0^{\pi/4} \cos 2x \, dx \qquad \text{identity: } \cos^2 x - \sin^2 x = \cos 2x$$

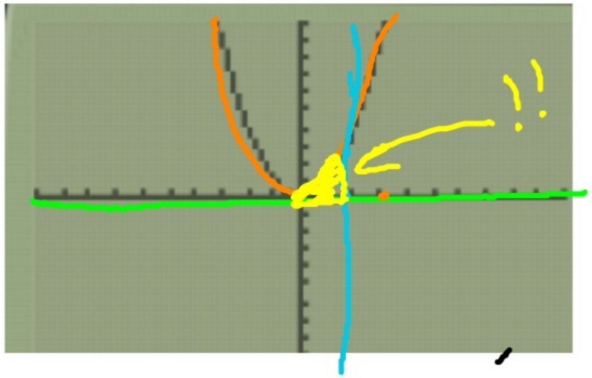
$$= \pi \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{\pi}{2} \text{ units cubed.}$$

Now try Exercise 17.

In Exercises 11–20, find the volume of the solid generated by revolving the region bounded by the lines and curves about the x-axis. dx

11. $y = x^2, \quad y = 0, \quad x = 2$ $32\pi/5$
12. $y = x^3, \quad y = 0, \quad x = 2$ $128\pi/7$
13. $y = \sqrt{9 - x^2}, \quad y = 0$ 36π
14. $y = x - x^2, \quad y = 0$ $\pi/30$
15. $y = x, \quad y = 1, \quad x = 0$ $2\pi/3$
16. $y = 2x, \quad y = x, \quad x = 1$ π
17. $y = x^2 + 1, \quad y = x + 3$ $117\pi/5$
18. $y = 4 - x^2, \quad y = 2 - x$ $108\pi/5$
19. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$ $\pi^2 - 2\pi$
20. $y = -\sqrt{x}, \quad y = -2, \quad x = 0$ 8π

"disc"



$$\pi \int_0^2 \left(x^2\right)^2 dx$$

$$\pi \int_0^2 x^4 \, dx$$

$$\pi \frac{x^5}{5} \Big|_0^2 = \frac{\pi(2)^5}{5}$$

In Exercises 11–20, find the volume of the solid generated by revolving the region bounded by the lines and curves about the x-axis.

11. $y = x^2, \quad y = 0, \quad x = 2$ $32\pi/5$

12. $y = x^3, \quad y = 0, \quad x = 2$ $128\pi/7$

13. $y = \sqrt{9 - x^2}, \quad y = 0$ 36π

14. $y = x - x^2, \quad y = 0$ $\pi/30$

15. $y = x, \quad y = 1, \quad x = 0$ $2\pi/3$

16. $y = 2x, \quad y = x, \quad x = 1$ π

17. $y = x^2 + 1, \quad y = x + 3$ $117\pi/5$

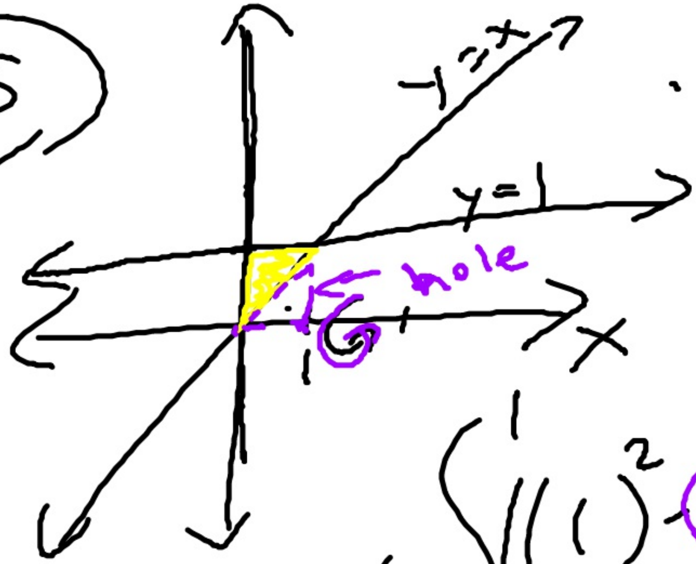
18. $y = 4 - x^2, \quad y = 2 - x$ $108\pi/5$

19. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$ $\pi^2 - 2\pi$

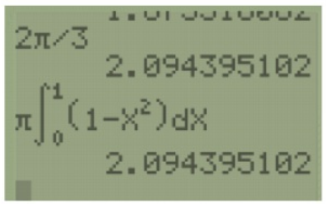
20. $y = -\sqrt{x}, \quad y = -2, \quad x = 0$ 8π

"washer"

15



$$\pi \int_0^1 \left((1)^2 - (x)^2\right) dx$$



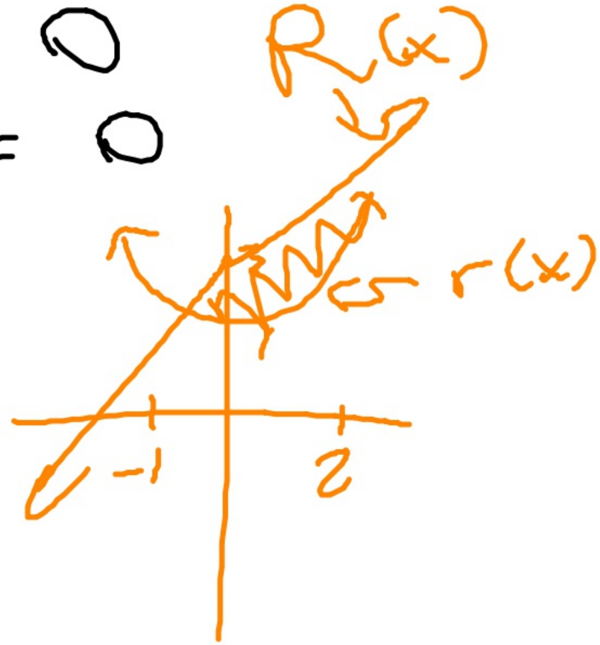
17. $y = x^2 + 1, \quad y = x + 3$ $117\pi/5$

$$x^2 + 1 = x + 3$$

$$x^2 - x - 2 = 0$$

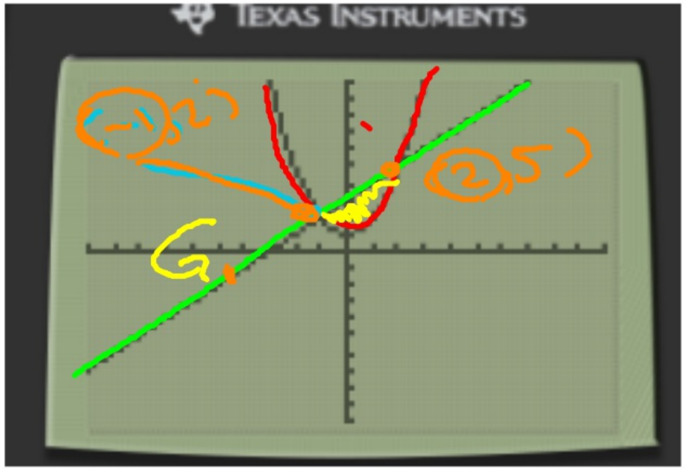
$$(x+1)(x-2) = 0$$

$$x = -1, 2$$



17. $y = x^2 + 1, \quad y = x + 3$

$117\pi/5$



πR^2

πr^2

$\pi R^2 - \pi r^2$

For intersections...

$\boxed{\text{2nd}} \rightarrow \text{calc} \rightarrow 5:$

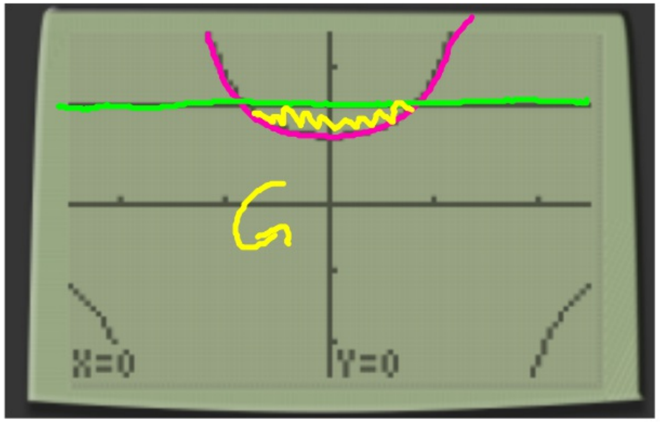
$$\pi \int_{-1}^{2} \left((x+3)^2 - (x^2+1)^2 \right) dx$$

19. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$ $\quad \pi^2 - 2\pi$

$$Y_1 = \frac{1}{\cos x}$$

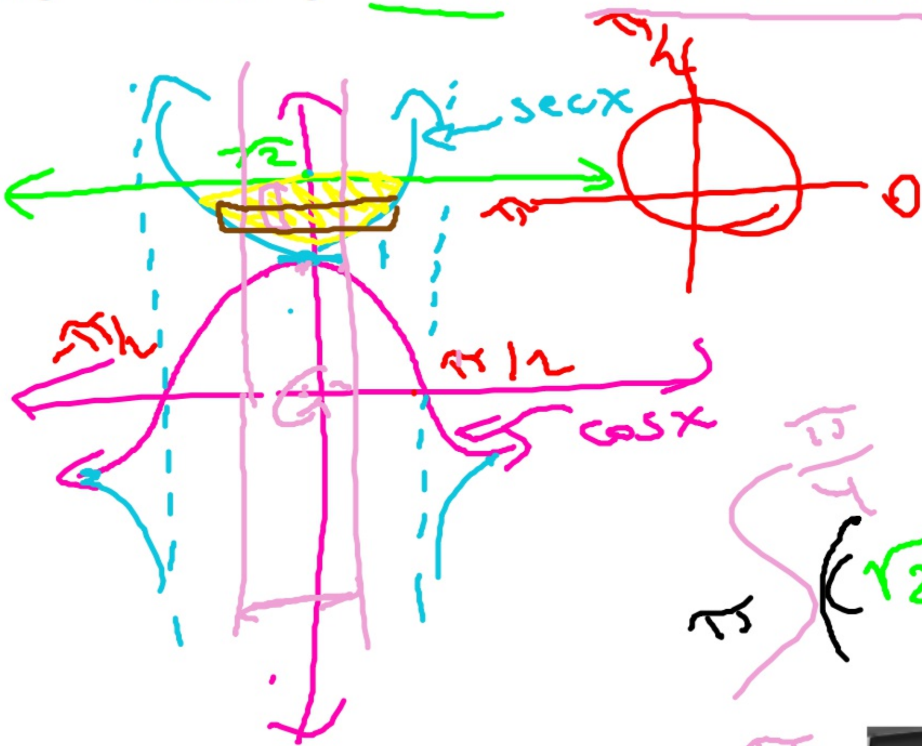
$$Y_2 = \sqrt{2}$$

$$\pi R^2 - \pi r^2$$



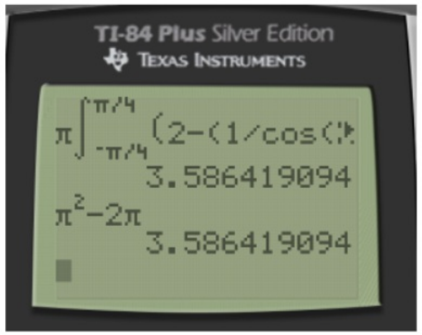
$$\pi \int_{-\pi/4}^{\pi/4} \left[\left(\sqrt{2}\right)^2 - (\sec x)^2 \right] dx$$

19. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4 \qquad \pi^2 - 2\pi$



$$\left(\frac{1}{\cos x}\right)$$

$$\pi \int_{-\pi/4}^{\pi/4} \left((\sqrt{2})^2 - (\sec x)^2 \right) dx$$



$$x^2 \longrightarrow \sqrt{x}$$

$$-\sqrt{x}$$

In Exercises 11–20, find the volume of the solid generated by revolving the region bounded by the lines and curves about the x-axis.

11. $y = x^2, \quad y = 0, \quad x = 2$ $32\pi/5$

12. $y = x^3, \quad y = 0, \quad x = 2$ $128\pi/7$

13. $y = \sqrt{9 - x^2}, \quad y = 0$ 36π

14. $y = x - x^2, \quad y = 0$ $\pi/30$

15. $y = x, \quad y = 1, \quad x = 0$ $2\pi/3$

16. $y = 2x, \quad y = x, \quad x = 1$ π

17. $y = x^2 + 1, \quad y = x + 3$ $117\pi/5$

18. $y = 4 - x^2, \quad y = 2 - x$ $108\pi/5$

19. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$ $\pi^2 - 2\pi$

20. $y = -\sqrt{x}, \quad y = -2, \quad x = 0$ 8π

EXAMPLE 4 Finding Volumes Using Cylindrical Shells

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

SOLUTION

1. Sketch the region and draw a line segment across it parallel to the axis of revolution (Figure 7.26). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). The width of the segment is the shell thickness dy. (We drew the shell in Figure 7.27, but you need not do that.)

"shell" $= 2\pi r h$

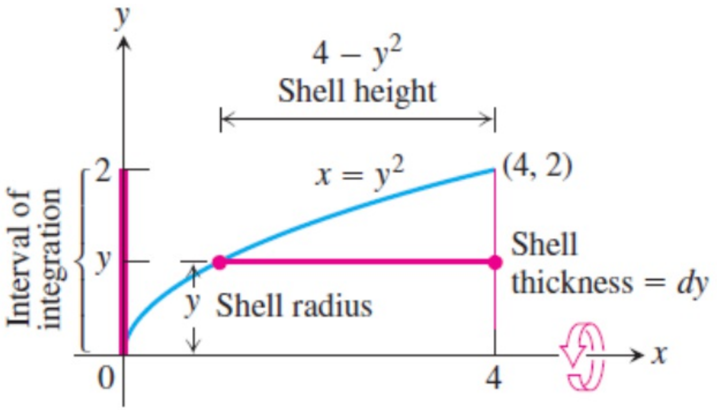


Figure 7.26 The region, shell dimensions, and interval of integration in Example 4.

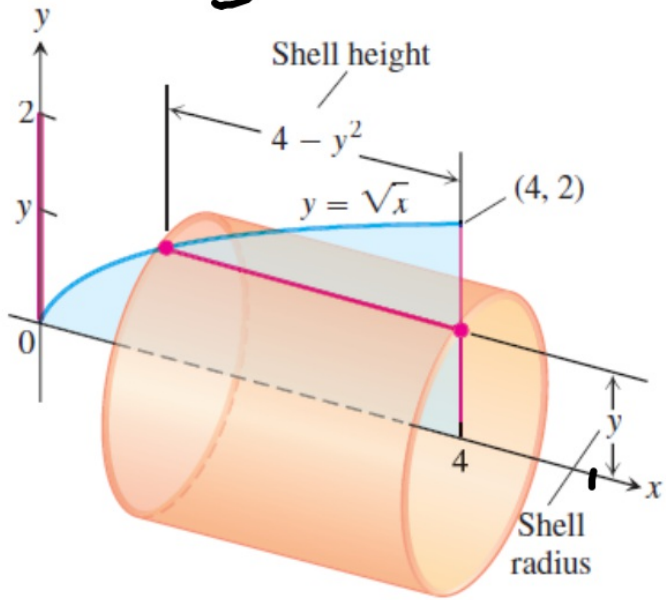


Figure 7.27 The shell swept out by the line segment in Figure 7.26.

2. Identify the limits of integration: y runs from 0 to 2.

3. Integrate to find the volume. $2\pi r h$

$$V = \int_0^2 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy$$

$$= \int_0^2 2\pi(y)(4 - y^2)\, dy = 8\pi$$

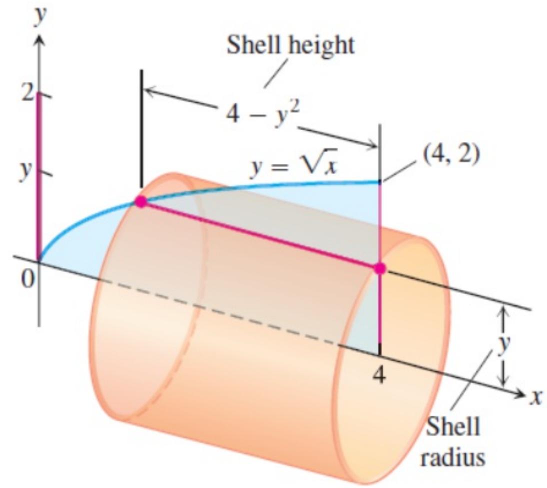


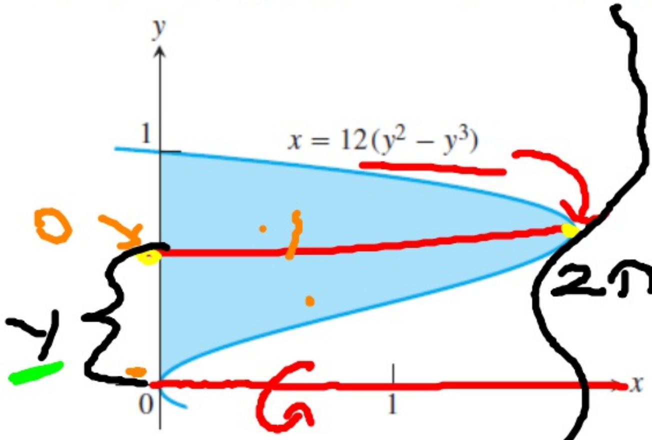
Figure 7.27 The shell swept out by the line segment in Figure 7.26.

Now try Exercise 33(a).

In Exercises 33 and 34, use the cylindrical shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis.

33. **(a)** the x-axis $6\pi/5$ **(b)** the line $y = 1$ $4\pi/5$

(c) the line $y = 8/5$ 2π **(d)** the line $y = -2/5$ 2π



$$2\pi \int_0^1 \left(\left[12(y^2-y^3)\right] - (0)\right)(y)\,dy$$

$$= 2\pi \int_0^1 \left[12y(y^2-y^3)\right] dy$$

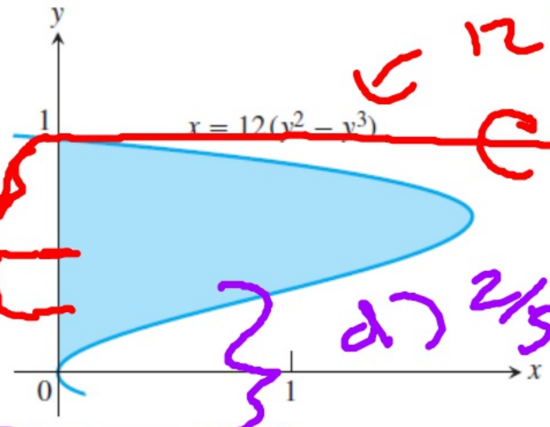
In Exercises 33 and 34, use the cylindrical shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis.

33. **(a)** the x-axis $6\pi/5$ **(b)** the line $y = 1$ $4\pi/5$

(c) the line $y = 8/5$ 2π **(d)** the line $y = -2/5$ 2π

$2\pi r h$

$12(y^2-y^3) = \text{height!}$



$y = 1$

radius?

y

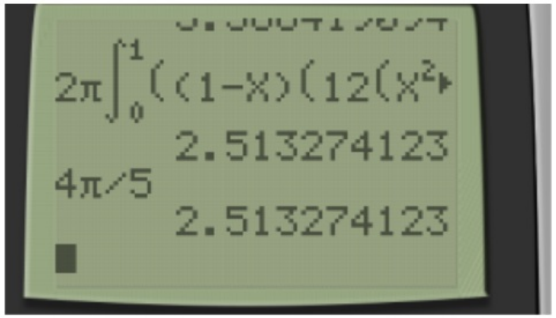
d) $\frac{2}{5} + y = \text{radius!}$

$1 - y = \text{radius!}$

$-\frac{2}{5}$

$\frac{8}{5} - y = \text{radius}$ (part c)

$$2\pi \int_0^1 \left[(1-y)\left(12(y^2-y^3)\right)\right] dy$$



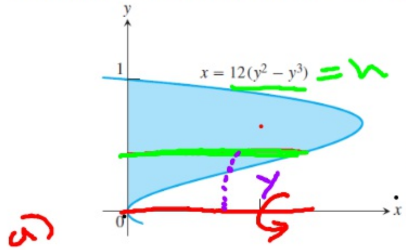
In Exercises 33 and 34, use the cylindrical shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis.

the challenge..

"shell" $= 2\pi r h$

33. **(a)** the x-axis $6\pi/5$ **(b)** the line $y = 1$ $4\pi/5$

(c) the line $y = 8/5$ 2π **(d)** the line $y = -2/5$ 2π



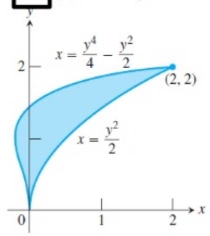
radius... $r = y$

$$2\pi \int_0^1 y(12(y^2-y^3))\,dy$$

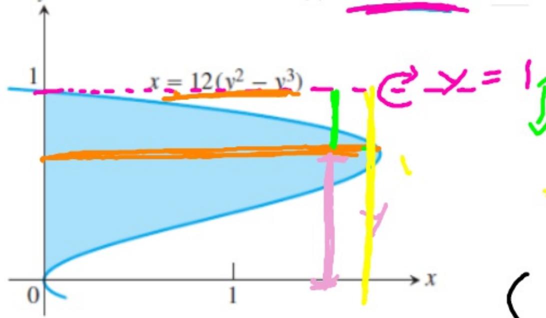
≈ 3.77

34. **(a)** the x-axis $\boxed{}$ **(b)** the line $y = 2$ $\boxed{}$

(c) the line $y = 5$ $\boxed{}$ **(d)** the line $y = -5/8$ $\boxed{}$



(b) the line $y = 1$ $4\pi/5$



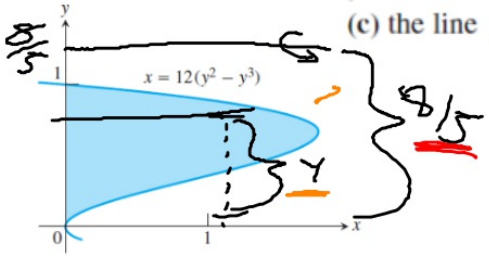
radius...?

$1 - y$

$$2\pi \int_0^1 (1-y)(12(y^2-y^3))\,dy$$

≈ 2.51

(c) the line $y = 8/5$ 2π

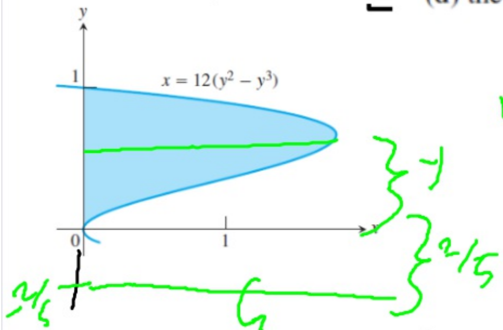


$r = \frac{8}{5} - y$

$$2\pi \int_0^1 \left(\frac{8}{5} - y\right)\left(12(y^2 - y^3)\right) dy$$

$2\pi r h$

(d) the line $y = -2/5$ 2π



$r = \left(\frac{2}{5} + y\right)$

$$2\pi \int_0^1 \left(\frac{2}{5} + y\right)\left(12(y^2 - y^3)\right) dy$$